Solutions of $\left(v_r - \frac{v_c^2}{v_r}\right)\frac{\partial v_r}{\partial r} = \frac{2\,v_c^2}{r} - \frac{G\,M_\odot}{r^2}$

$\frac{\partial v_r}{\partial r} \Rightarrow \frac{d v_r}{d r}$

integrate with r

$$\int \left(v_r - \frac{v_c^2}{v_r}\right) d v_r = \int \left(\frac{2\,v_c^2}{r} - \frac{G\,M_\odot}{r^2}\right) dr$$

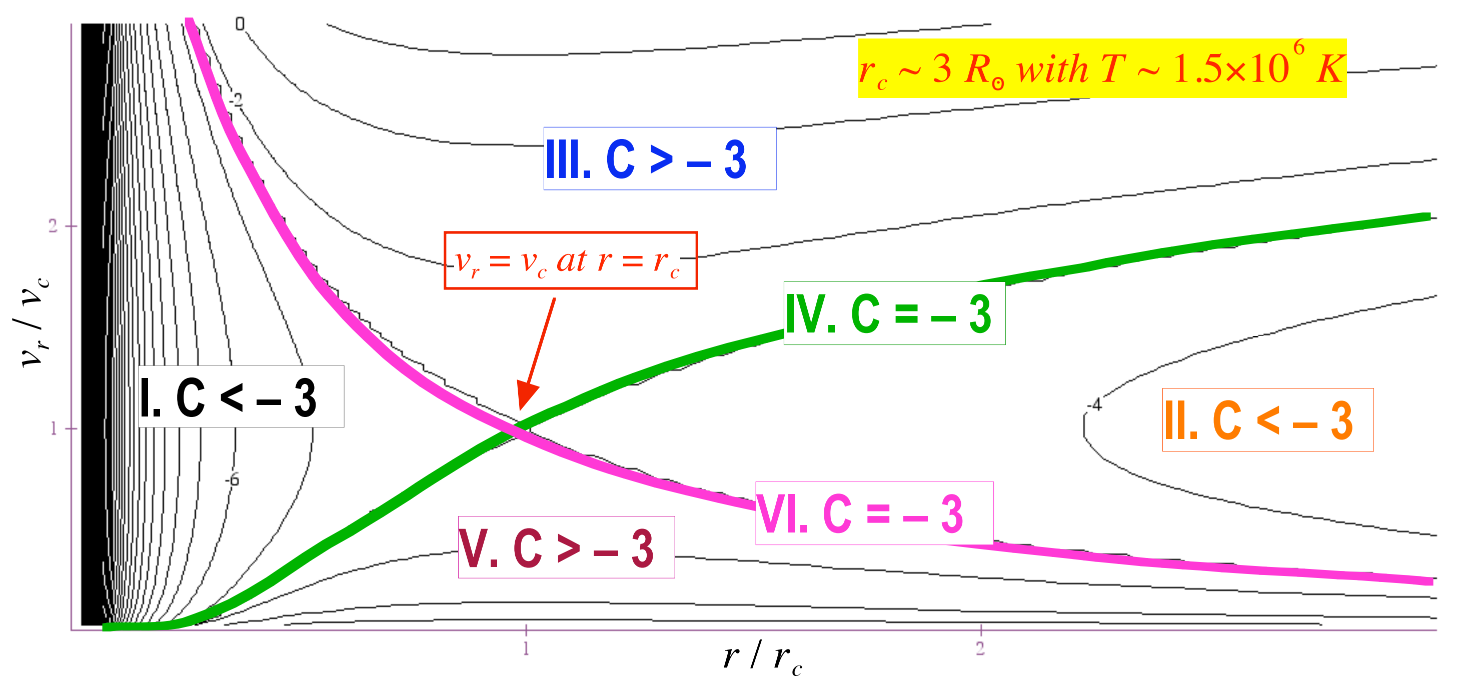
dimensionless form

$$\left(\frac{v_r}{v_c}\right)^2 - \ln\left(\frac{v_r}{v_c}\right)^2 = 4\ln\frac{r}{r_c} + 4\left(\frac{r}{r_c}\right)^{-1} + C$$

$r_c \equiv \frac{G\,M_\odot}{2\,v_c^2}$: **critical radius**

$C = constant$

$C = -3$ when $(r, v_r) = (r_c, v_c)$.



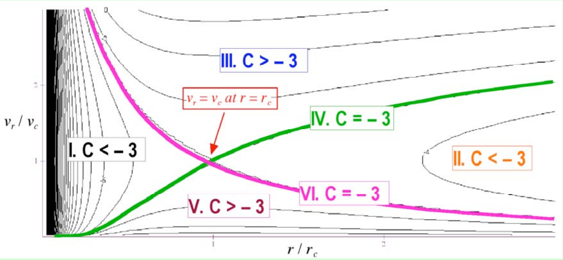
$r_c \sim 3\,R_\odot$ with $T \sim 1.5\times10^6\,K$

$v_r = v_c$ at $r = r_c$

Properties of the solutions...

I. C < – 3 **II. C < – 3**

These two solutions are **physically incorrect**
(although they are correct mathematically)
because flow velocity takes two different values at the same position.



III. C > – 3 **IV. C = – 3** **V. C > – 3** **VI. C = – 3**

These four solutions are physically correct. Which is appropriate for solar wind?

III. C > – 3 **VI. C = – 3** ... starts with a supersonic flow ($v_r > v_c$) at a solar surface
=> inconsistent with inner boundary condition

IV. C = – 3 ... starts with a subsonic flow ($v_r < v_c$) at a solar surface, and becomes supersonic
=> consistent with inner boundary condition

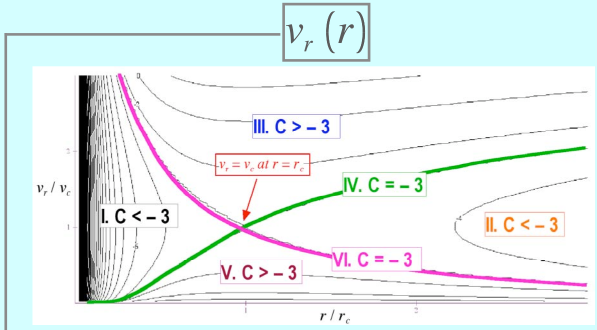
V. C > – 3 ... starts with a subsonic flow ($v_r < v_c$) at a solar surface, and remains subsonic
=> consistent with inner boundary condition

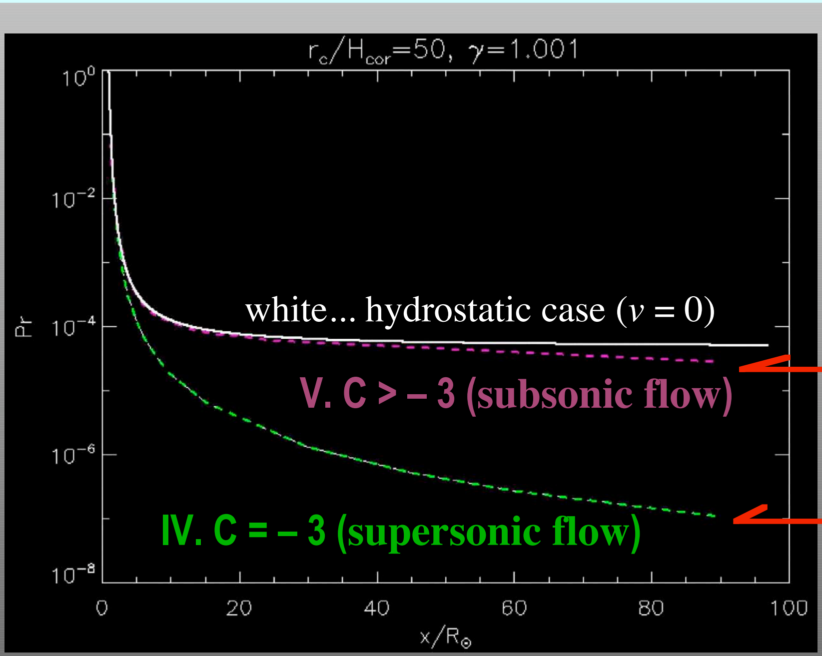
Outer boundary condition (solution IV vs. solution V)...

We focus on **gas pressure distribution**:

$$p(r) = \rho(r)\frac{k_B T}{\bar{m}} = \rho(r)\, v_c^2\ , \quad (v_c \textit{ is constant})$$

$$\rho(r) = \frac{\rho(R_\odot)\, v_r(R_\odot)\, R_\odot^2}{v_r(r)\, r^2}$$





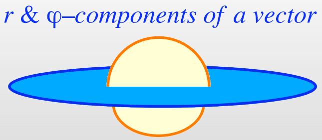
The value of gas pressure at outer boundary is different between IV (**supersonic case, low pressure**) and V (**subsonic case, high pressure**).

↓

Observation suggests a very low value of gas pressure at the outer boundary ($P^{interstellar} \sim 10^{-17} P^{solar\ surface}$), so we select solution IV (**supersonic case**).

Magnetohydrodynamic model

Weber & Davis 1.5-dimensional model



(axisymmetric, on the equatorial plane, polytropic, steady, rotation)

(depends on r, *r & φ–components of a vector* are considered)

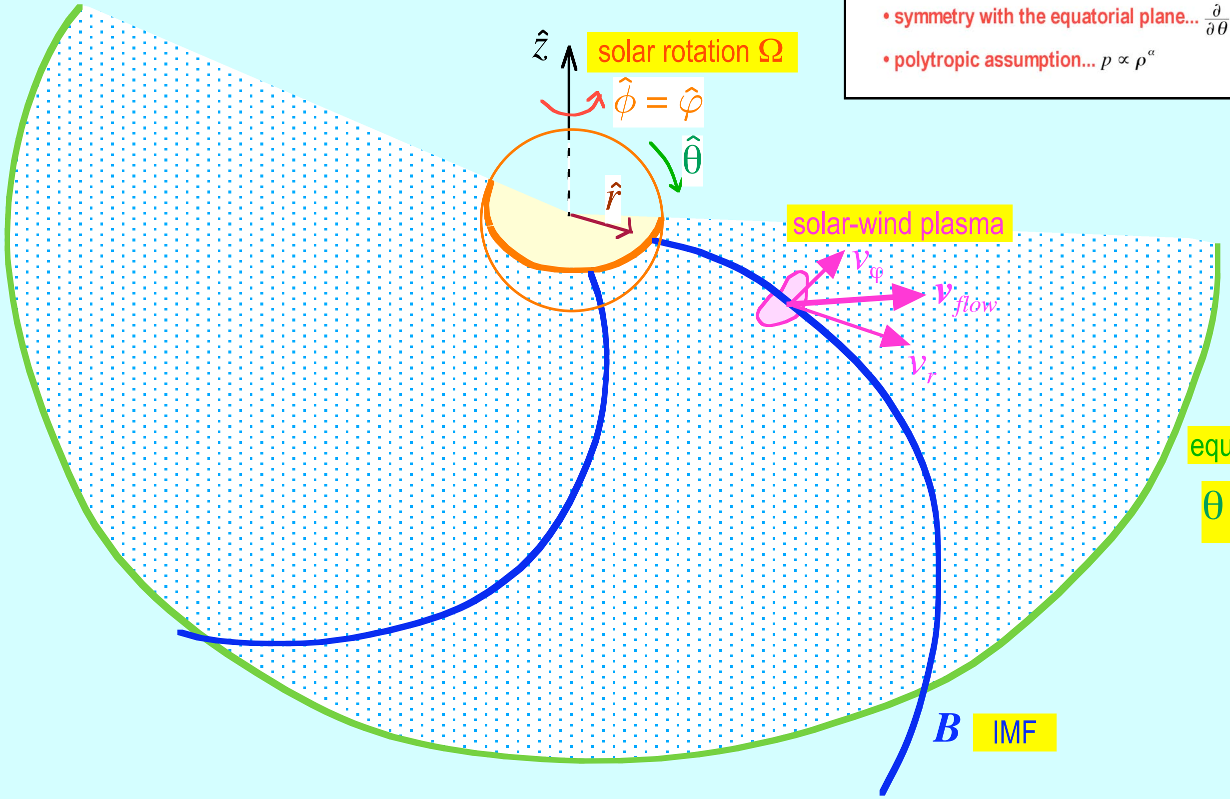
(r, θ, ϕ)... spherical coordinates with θ = π/2

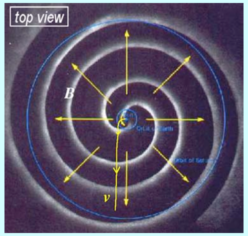
$$\boldsymbol{v}(r,\theta,\phi) = v_r(r,\theta,\phi)\,\hat{\boldsymbol{r}} + v_\theta(r,\theta,\phi)\,\hat{\theta} + v_\phi(r,\theta,\phi)\,\hat{\phi}$$

$$\longrightarrow v_r\left(r,\theta=\frac{\pi}{2}\right)\hat{\boldsymbol{r}} + v_\phi\left(r,\theta=\frac{\pi}{2}\right)\hat{\theta}$$

Assumption:

- axial symmetry... $\frac{\partial}{\partial \varphi} = 0$
- steady state... $\frac{\partial}{\partial t} = 0$
- only equatorial plane is considered... $\theta = \frac{\pi}{2}$
- symmetry with the equatorial plane... $\frac{\partial}{\partial \theta} = 0 \; at \; \theta = \frac{\pi}{2}$, $v_\theta = 0$, $B_\theta = 0$
- polytropic assumption... $p \propto \rho^\alpha$





Basic equations (differential form):

$\boxed{\text{CGS unit}}$ $\quad \dfrac{\partial}{\partial r} \Rightarrow \dfrac{d}{dr}$

mass conservation...

$$\nabla \bullet (\rho\, \boldsymbol{v}) = 0 \quad \xrightarrow{\qquad\qquad} \quad \frac{1}{r^2}\frac{d}{d\,r}\left(r^2 \rho\, v_r\right) = 0$$

momentum equatoin...

$$\rho\, \boldsymbol{v}\bullet\nabla\, \boldsymbol{v} = -\, \nabla p + \frac{1}{4\pi}\left(\nabla \times \boldsymbol{B}\right) \times \boldsymbol{B} - \frac{G\, M_\odot\, \rho}{r^2}\, \hat{\boldsymbol{r}} = 0$$

$$\xrightarrow[\text{r-component}]{\qquad\qquad} \quad \rho\, v_r \frac{dv_r}{dr} - \frac{\rho\, v_\varphi^2}{r} = -\frac{dp}{dr} - \frac{1}{4\pi}\frac{B_\varphi}{r}\frac{d}{dr}\left(r\, B_\varphi\right) - \frac{G\, M_\odot\, \rho}{r^2}$$

$$\xrightarrow[\phi\text{-component}]{\qquad\qquad} \quad \rho\, v_r \frac{d}{dr}\left(r\, v_\varphi\right) = \frac{1}{4\pi} B_r \frac{d}{dr}\left(r\, B_\varphi\right)$$

energy equation...

$$\boldsymbol{v}\bullet\nabla\left(\frac{p}{\rho^\alpha}\right) = 0 \quad \xrightarrow{\qquad\qquad} \quad v_r \frac{d}{dr}\left(\frac{p}{\rho^\alpha}\right) = 0$$

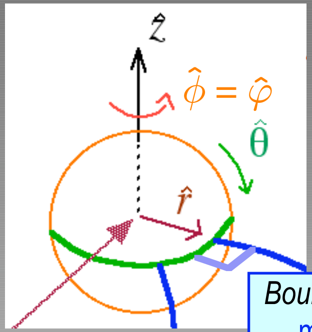
induction equation...

$$\nabla \times \left(\boldsymbol{v} \times \boldsymbol{B}\right) = \boldsymbol{0} \quad \xrightarrow[\theta\text{-component}]{\qquad\qquad} \quad \frac{d}{d\,r}\left(r\left[v_\varphi\, B_r - v_r\, B_\varphi\right]\right) = 0$$

magnetic flux conservation...

$$\nabla \bullet \boldsymbol{B} = 0 \quad \xrightarrow{\qquad\qquad} \quad \frac{1}{r^2}\frac{d}{d\,r}\left(r^2\, B_r\right) = 0$$

Basic equations (integral form):



Boundary condition:
magnetic field is **normal to** solar surface

mass conservation… $r^2 \rho v_r = f$ (constant)

energy eqution… $\dfrac{p}{\rho^{\alpha}} = K$ (constant)

induction equation… $v_\varphi B_r - v_r B_\varphi = \dfrac{C_0}{r} = \dfrac{\Omega R_\odot^2 B_0}{r}$

at $r = R_\odot, v_\varphi = \Omega R_\odot, B_r = B_0, B_\varphi = 0 \Rightarrow C_0 = \Omega R_\odot^2 B_0$ (constant)

magnetic flux conservation… $r^2 B_r = \Phi = R_\odot^2 B_0$ (constant)

$$v_\varphi B_r - v_r B_\varphi = \Omega r B_r$$

∫ ϕ-component of momentum × *dr*

=> angular momentum conservation (per unit mass)

$$\rho v_r \frac{d}{dr}\left(r v_\varphi\right) = \frac{1}{4\pi} B_r \frac{d}{dr}\left(r B_\varphi\right)$$

$$\frac{\rho v_r}{B_r} = \frac{f / r^2}{\Phi / r^2} = \frac{f}{\Phi} = const.$$

$r^2 \rho v_r = f$

$r^2 B_r = \Phi$

integrate with *r*

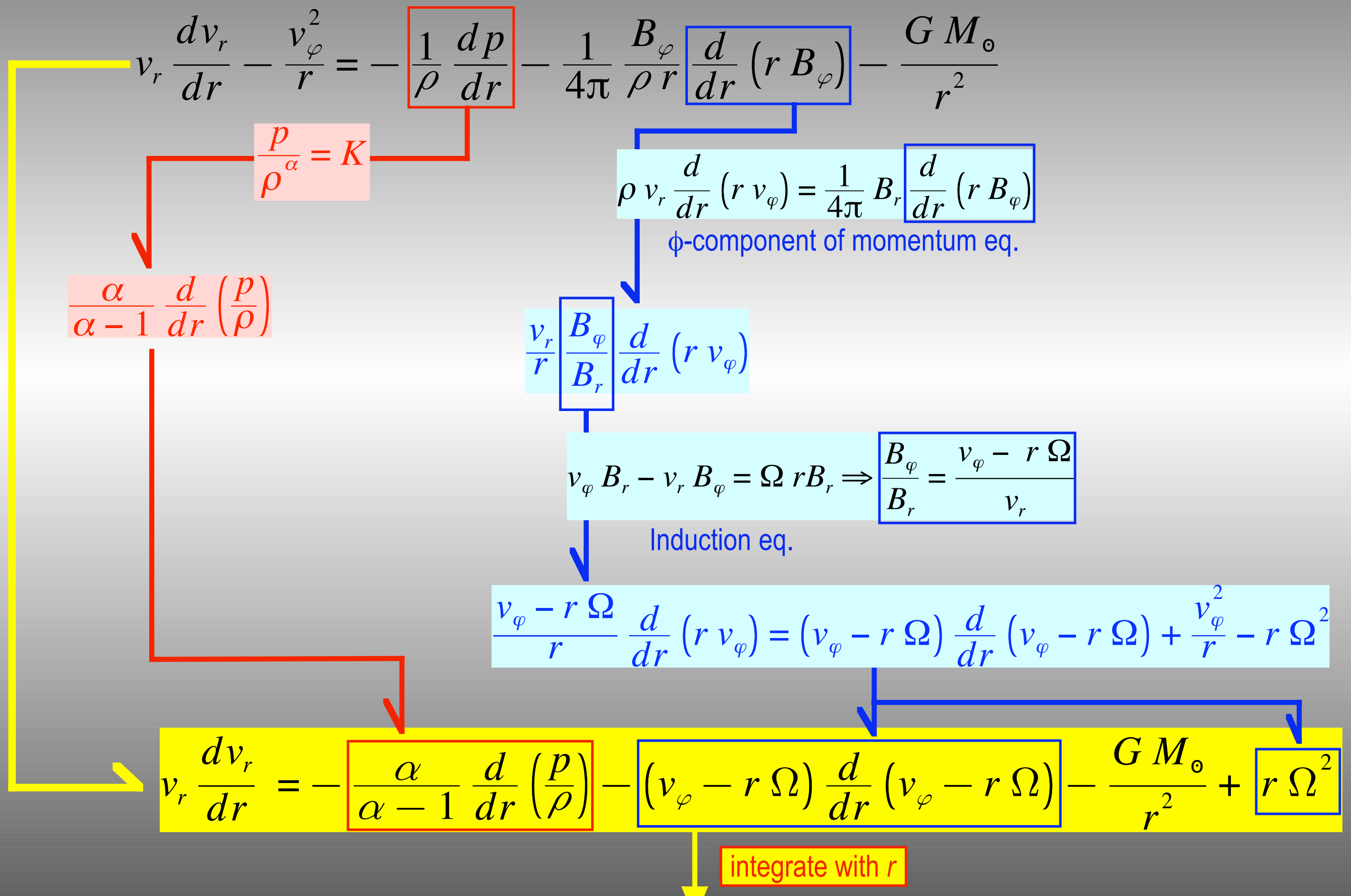
$$r v_\varphi - \frac{B_r}{4 \pi \rho v_r} r B_\varphi = L$$

... angular momentum per unit mass (constant)

$$L = v_\varphi\left(r = R_\odot\right) R_\odot = \Omega R_\odot R_\odot$$

Angular momentum conservation (per unit mass)

∫ *r*-component of momentum • *dr* => energy conservation (per unit mass)



$$v_r \frac{dv_r}{dr} - \frac{v_\varphi^2}{r} = -\boxed{\frac{1}{\rho}\frac{dp}{dr}} - \frac{1}{4\pi}\frac{B_\varphi}{\rho r}\boxed{\frac{d}{dr}\left(r B_\varphi\right)} - \frac{G M_\odot}{r^2}$$

$$\frac{p}{\rho^\alpha} = K$$

$$\frac{\alpha}{\alpha - 1}\frac{d}{dr}\left(\frac{p}{\rho}\right)$$

$$\rho\, v_r \frac{d}{dr}\left(r v_\varphi\right) = \frac{1}{4\pi} B_r \boxed{\frac{d}{dr}\left(r B_\varphi\right)}$$

ϕ-component of momentum eq.

$$\frac{v_r}{r}\boxed{\frac{B_\varphi}{B_r}}\frac{d}{dr}\left(r v_\varphi\right)$$

$$v_\varphi B_r - v_r B_\varphi = \Omega\, r B_r \Rightarrow \boxed{\frac{B_\varphi}{B_r} = \frac{v_\varphi - r\,\Omega}{v_r}}$$

Induction eq.

$$\frac{v_\varphi - r\,\Omega}{r}\frac{d}{dr}\left(r v_\varphi\right) = \left(v_\varphi - r\,\Omega\right)\frac{d}{dr}\left(v_\varphi - r\,\Omega\right) + \frac{v_\varphi^2}{r} - r\,\Omega^2$$

$$v_r \frac{dv_r}{dr} = -\boxed{\frac{\alpha}{\alpha - 1}\frac{d}{dr}\left(\frac{p}{\rho}\right)} - \boxed{\left(v_\varphi - r\,\Omega\right)\frac{d}{dr}\left(v_\varphi - r\,\Omega\right)} - \frac{G M_\odot}{r^2} + \boxed{r\,\Omega^2}$$

integrate with *r*

Energy conservation (per unit mass)

$$\overbrace{\frac{1}{2} v_r^2}^{\text{radial flow kinetic energy}} + \underbrace{\frac{1}{2}\left(v_\varphi - r\,\Omega\right)^2}_{\text{rotational energy}} + \underbrace{\frac{\alpha}{\alpha - 1}\frac{p}{\rho}}_{\text{thermal energy}} - \overbrace{\frac{G M_\odot}{r}}^{\text{gravitational energy}} - \underbrace{\frac{1}{2}\left(r\,\Omega\right)^2}_{\text{centrifugal energy}} = E$$

... energy per unit mass (constant)