

Role of Magnetic Field

It drives evolution of flow velocity.

$$\rho \frac{d\mathbf{v}}{dt} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

Magnetic field in MHD

=> two components of $\mathbf{j} \times \mathbf{B}$ force (magnetic tension & magnetic pressure)

Decompose $\mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}$ into two components:

A formula of vector differentiation:

$$\nabla (\mathbf{X} \cdot \mathbf{Y}) = (\mathbf{Y} \cdot \nabla) \mathbf{X} + (\mathbf{X} \cdot \nabla) \mathbf{Y} + \mathbf{Y} \times (\nabla \times \mathbf{X}) + \mathbf{X} \times (\nabla \times \mathbf{Y})$$

Substitute $\mathbf{X} = \mathbf{B}$, $\mathbf{Y} = \mathbf{B}$ into this formula, then we have

$$\nabla (\mathbf{B} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{B})$$

This is transformed into

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left(\frac{B^2}{2} \right)$$

$$\mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$= \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left(\frac{B^2}{2\mu_0} \right)$$

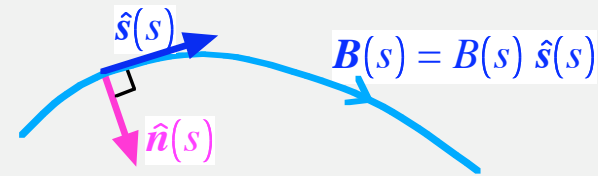
tension part

pressure part

Magnetic tension: $F_T = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B}$

$$\begin{aligned}
 (\mathbf{B} \cdot \nabla) \mathbf{B} &= B \hat{\mathbf{s}} \cdot \nabla (B \hat{\mathbf{s}}) \\
 &= B \frac{\partial}{\partial s} (B \hat{\mathbf{s}}) \\
 &= \left(B \frac{\partial B}{\partial s} \right) \hat{\mathbf{s}} + B^2 \frac{\partial \hat{\mathbf{s}}}{\partial s} \\
 &= \frac{\partial}{\partial s} \left(\frac{1}{2} B^2 \right) \hat{\mathbf{s}} + B^2 \frac{\hat{\mathbf{n}}}{R_c}
 \end{aligned}$$

$\hat{\mathbf{s}}(s)$... tangential vector along a magnetic field line (unit vector)

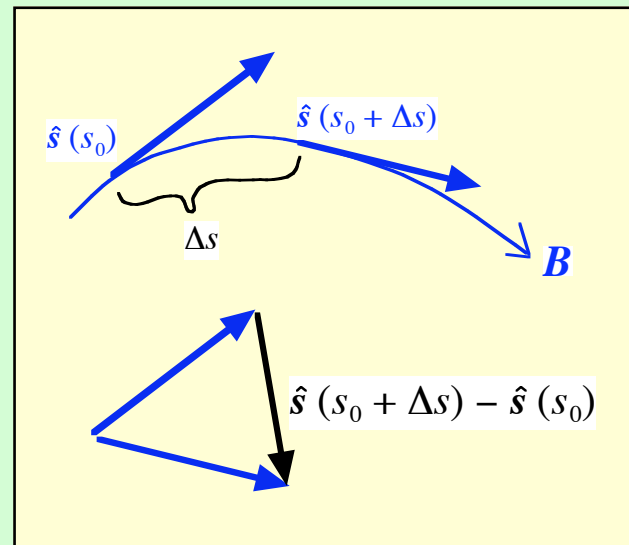
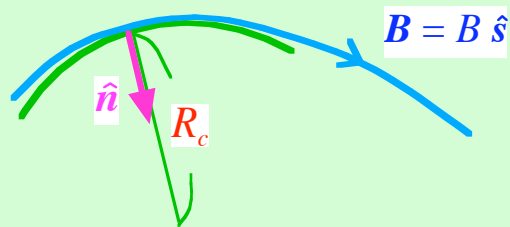


$$\hat{\mathbf{s}} \cdot \frac{\partial \hat{\mathbf{s}}}{\partial s} = \frac{1}{2} \frac{\partial}{\partial s} (\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}) = \frac{1}{2} \frac{\partial}{\partial s} (1) = 0$$

$$\Rightarrow \hat{\mathbf{s}} \perp \frac{\partial \hat{\mathbf{s}}}{\partial s} \Rightarrow \frac{\partial \hat{\mathbf{s}}}{\partial s} \parallel \hat{\mathbf{n}} \text{ normal vector (unit vector)}$$

$$\begin{aligned}
 \frac{\partial \hat{\mathbf{s}}}{\partial s} &= \lim_{\Delta s \rightarrow 0} \frac{\hat{\mathbf{s}}(s_0 + \Delta s) - \hat{\mathbf{s}}(s_0)}{\Delta s} \\
 &= \kappa \hat{\mathbf{n}} = \frac{\hat{\mathbf{n}}}{R_c}
 \end{aligned}$$

κ ... curvature, $R_c \equiv \frac{1}{\kappa}$... curvature radius

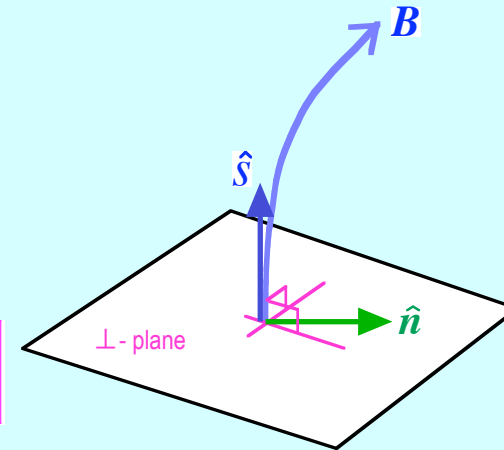


$$\mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$= \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla \left(\frac{B^2}{2\mu_0} \right)$$

$$\frac{\partial}{\partial s} \left(\frac{B^2}{2\mu_0} \right) \hat{s} + \frac{B^2}{\mu_0} \frac{\hat{n}}{R_c} - \left[\frac{\partial}{\partial s} \left(\frac{B^2}{2\mu_0} \right) \hat{s} + \nabla_{\perp} \left(\frac{B^2}{2\mu_0} \right) \right]$$

canceled



$$\mathbf{j} \times \mathbf{B} = \frac{B^2}{\mu_0} \frac{\hat{n}}{R_c} - \nabla_{\perp} \left(\frac{B^2}{2\mu_0} \right)$$

magnetic
tension force

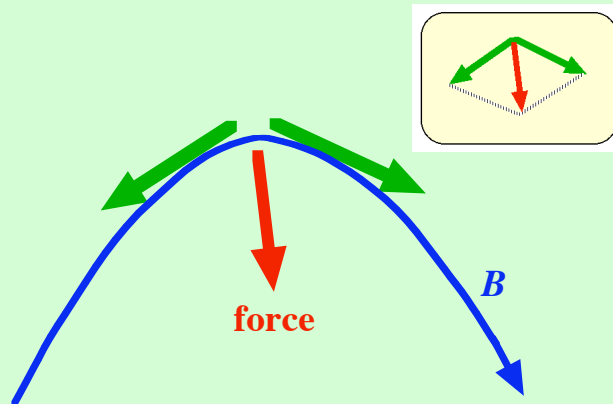
magnetic pressure
gradient force

Both **magnetic tension force** and **magnetic pressure gradient force** work in the direction **perpendicular to** magnetic field.

Features of magnetic tension and magnetic pressure

Magnetic tension force

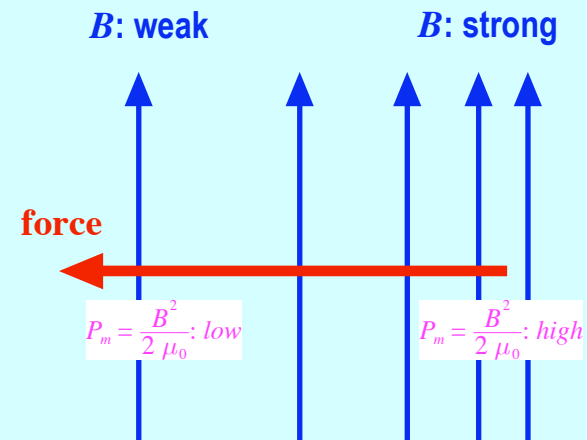
... depends on **shape of a field line**



$$\frac{B^2}{\mu_0} \frac{\hat{n}}{R_c}$$

Magnetic pressure gradient force

... depends on **distribution of field strength**

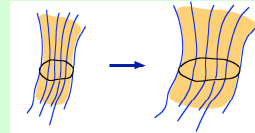


$$-\nabla_{\perp} \left(\frac{B^2}{2\mu_0} \right)$$

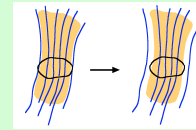
Summary of the roles of \mathbf{E}^{MHD} and \mathbf{B} in MHD

Electric field => drives evolution of magnetic field through $\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}^{MHD}$

$$-\nabla \times \mathbf{E}_{conv} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

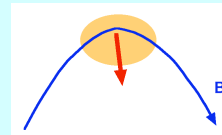


$$-\nabla \times \mathbf{E}_{resis} = -\nabla \times (\eta_{diff} \nabla \times \mathbf{B})$$

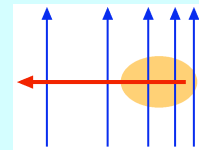


Magnetic Field => drives evolution of flow velocity through $\rho \frac{d\mathbf{v}}{dt} = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}$

$$\mathbf{F}_t = \frac{B^2}{\mu_0} \frac{\hat{\mathbf{n}}}{R_c}$$



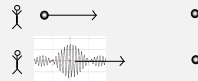
$$\mathbf{F}_m = -\nabla_{\perp} \left(\frac{B^2}{2\mu_0} \right)$$



MHD Wave

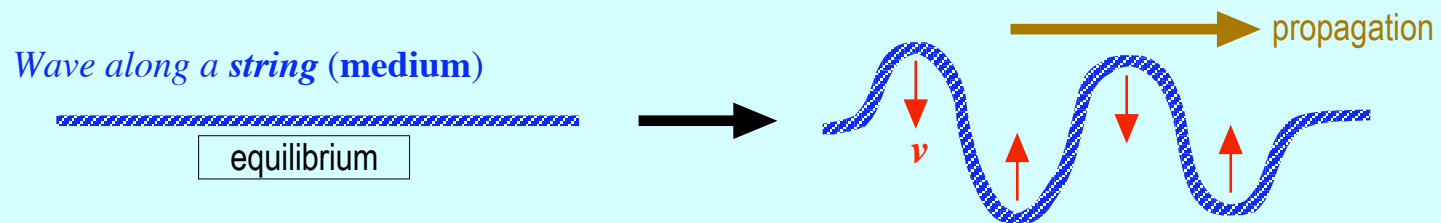
What is a wave?

It is a **time-dependent phenomenon** in which a **physical state** is transported from one location to other locations through a **medium** (what is transported is **physical state**, not **physical object**).



Two types of waves

Transverse wave... oscillation direction \perp propagation direction



Longitudinal wave... oscillation direction \parallel propagation direction

