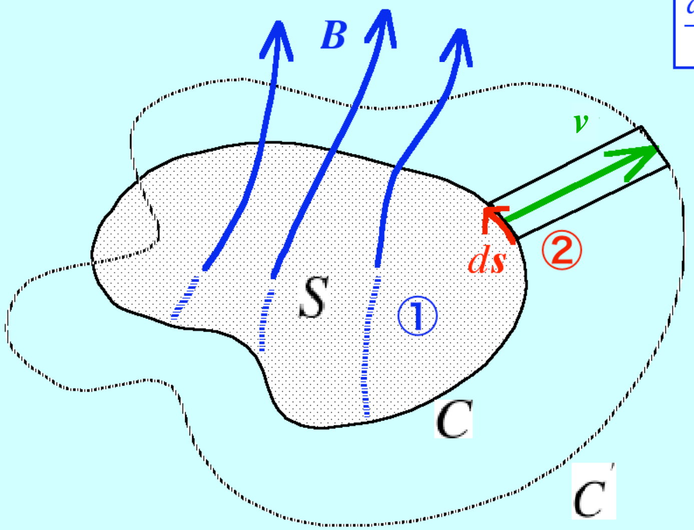
Evolution of magnetic field via $\boldsymbol{E}_{conv} \equiv -\boldsymbol{v} \times \boldsymbol{B}$:

$$\Phi_S = \iint_S \boldsymbol{B} \bullet d\boldsymbol{S}$$

... total magnetic flux crossing S



Lagrangian derivative

Change of B when C is fixed ①
\+
Change of C when B is fixed ②

per unit time

$$\frac{d\Phi_S}{dt} = \iint_S \frac{\partial \boldsymbol{B}}{\partial t} \bullet d\boldsymbol{S} + \oint_C \boldsymbol{B} \bullet (\boldsymbol{v} \times d\boldsymbol{s})$$

scalar triple vector product

$$= \iint_S \frac{\partial \boldsymbol{B}}{\partial t} \bullet d\boldsymbol{S} + \oint_C (\boldsymbol{B} \times \boldsymbol{v}) \bullet d\boldsymbol{s}$$

Stokes's theorem

$$= \iint_S \frac{\partial \boldsymbol{B}}{\partial t} \bullet d\boldsymbol{S} + \iint_S \nabla \times (\boldsymbol{B} \times \boldsymbol{v}) \bullet d\boldsymbol{S}$$

$$= \iint_S \left[\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{B} \times \boldsymbol{v})\right] \bullet d\boldsymbol{S}$$

$$= \iint_S \left[\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B})\right] \bullet d\boldsymbol{S}$$

$$= 0$$

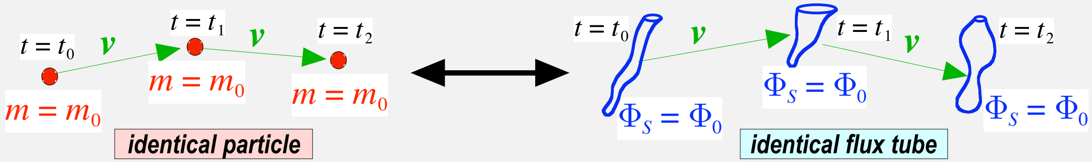
$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \Rightarrow \frac{d\Phi_S}{dt} = 0$$

Physical meaning of $\frac{d\Phi_S}{dt} = 0$ => Frozen-in evolution

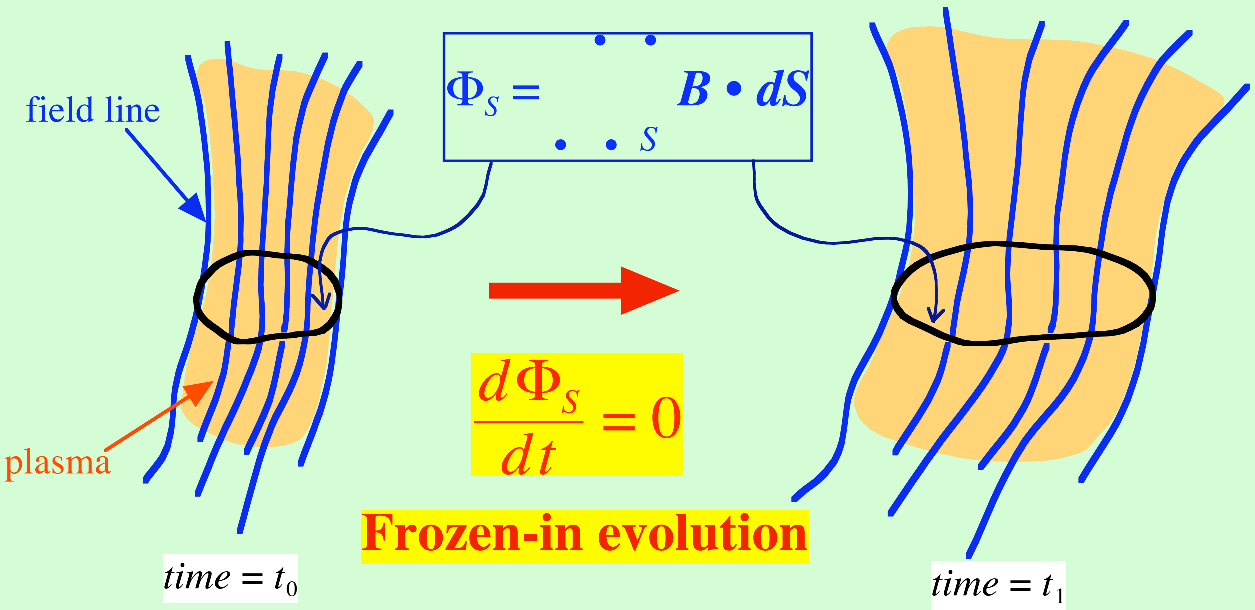
The number of magnetic field lines passing through the area bounded by any closed curve that moves together with a plasma **does not change with time**.

⇓

We can determine an **identical flux tube** in which magnetic field lines **are frozen into** the plasma.



Flux tube **keeps identity** while evolving *(even though its shape significantly changes…)*



Diffusion term: $-\nabla \times \boldsymbol{E}_{resis} = -\nabla \times \left(\eta_{diff} \nabla \times \boldsymbol{B}\right)$

Diffusive limit...

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left(\boldsymbol{E}_{conv} + \boldsymbol{E}_{resis}\right)$$

$$\frac{E_{resis}}{E_{conv}} \sim \frac{1}{\mathrm{R}_m}$$

$\mathrm{R_m} \not\gg 1$ → we cannot neglect diffusion term

Three conditions for ***diffusion limit*** ($\mathrm{R_m} \equiv \dfrac{v_0 l_0}{\eta_{diff}} << 1$):

I. typical length is **very small**

II. typical velocity is **very small**

III. magnetic diffusivity is **large**

In astronomical plasmas, their typical length l_0 is **usually very large**, so $\mathrm{R_m} \not\gg 1$ is **rare** except for a **region where magnetic field changes sharply** (e.g. *current sheet*).
This **significantly reduces** l_0 and may **enhance effective** η_{diff} via **kinetic processes** (e.g. *wave-particle interaction*).

Evolution of magnetic field via $\boldsymbol{E}_{resis} \equiv \eta_{diff} \nabla \times \boldsymbol{B}$:

When η_{diff} *is constant,* $\dfrac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left(\eta_{diff} \nabla \times \boldsymbol{B}\right) \Rightarrow \dfrac{\partial \boldsymbol{B}}{\partial t} = \eta_{diff} \nabla^2 \boldsymbol{B}$

$$\frac{\partial q}{\partial t} = \eta_{diff} \nabla^2 q \quad \textbf{... diffusion equation}$$

$q = T,\ j_x,\ B_y,\ \ldots$

Time scale *of diffusion*

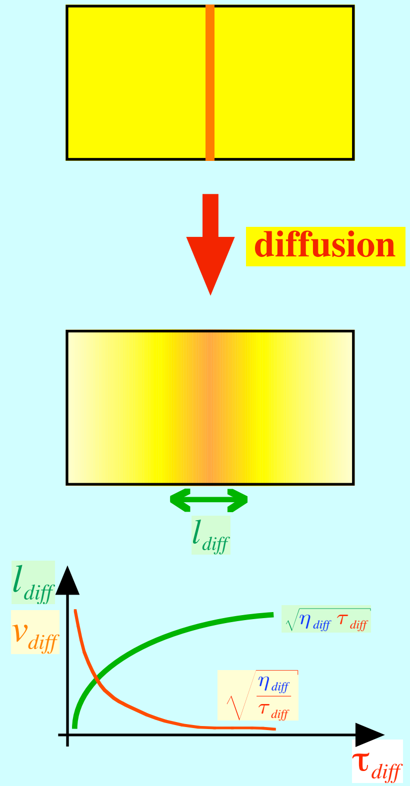
$$\frac{\partial q}{\partial t} = \eta_{diff} \nabla^2 q \Rightarrow \frac{q}{\tau_{diff}} \sim \eta_{diff} \frac{q}{l_{diff}^2} \quad \rightarrow \quad \tau_{diff} \sim \frac{l_{diff}^2}{\eta_{diff}}$$

Length scale *of diffusion*

$$l_{diff} \sim \sqrt{\eta_{diff} \tau_{diff}}$$

Velocity scale *of diffusion*

$$v_{diff} \sim \frac{l_{diff}}{\tau_{diff}} = \frac{\sqrt{\eta_{diff} \tau_{diff}}}{\tau_{diff}} = \sqrt{\frac{\eta_{diff}}{\tau_{diff}}}$$



Diffusion proceeds **fast at the beginning** and **then slowly**.

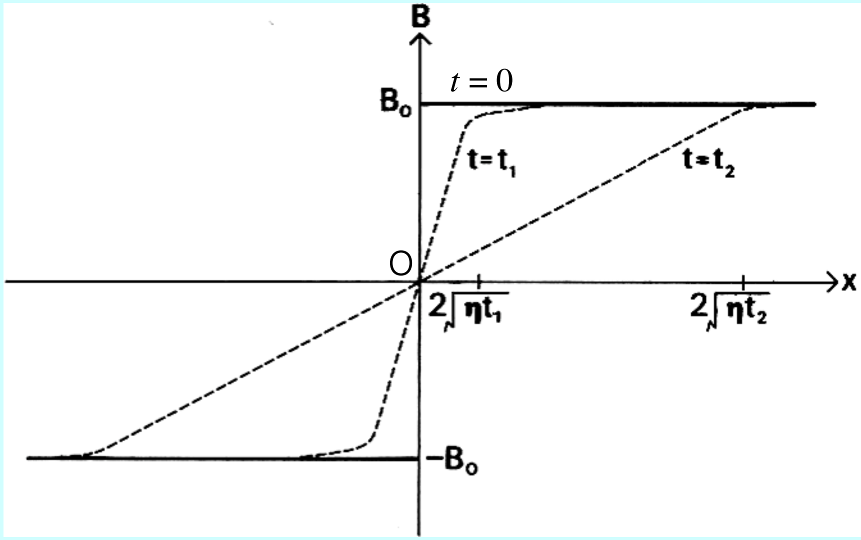
Diffusion of an antiparallel magnetic field (annihilation)

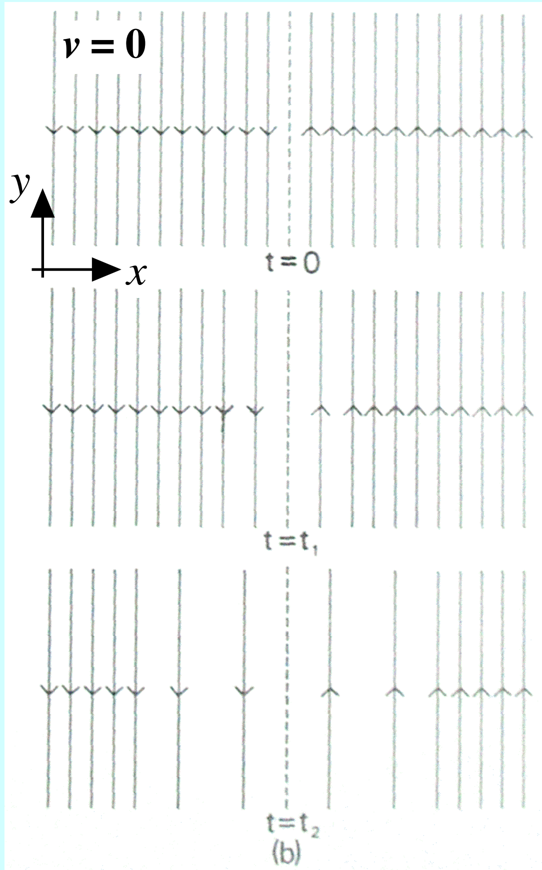
$$\frac{\partial B_y}{\partial t} = \eta_{diff} \frac{\partial^2}{\partial x^2} B_y$$

$B_y(x, t=0) = \begin{cases} B_0 & for\ x > 0 \\ -B_0 & for\ x < 0 \end{cases}$ ... initial condition

$B_y(x = \pm\infty, t) = \begin{cases} B_0 & for\ x = \infty \\ -B_0 & for\ x = -\infty \end{cases}$ ... boundary condition

$$B_y(x,t) = \frac{2\,B_0}{\sqrt{\pi}} \operatorname{erf}\left(\frac{x}{\sqrt{4\,\eta_{diff}\,t}}\right), \quad \operatorname{erf}(\xi) = \int_0^{\xi} e^{-u^2}\,du$$





Solar MHD (Priest 1982)

- magnetic field is annihilated at x = 0
- magnetic field diffuses through a static plasma => **violates frozen-in evolution**

Diffusion of ***B***… **non-fronzen-in evolution**