

Equation of state (fully ionized)

$$P^p = n k_B T^p \quad \dots \text{proton's equation of state}$$

$$P^e = n k_B T^e \quad \dots \text{electron's equation of state}$$

summation

$$P^{MHD} = 2 n k_B T^{MHD} = 2 \frac{\rho^{MHD}}{M + m} k_B T^{MHD} = \rho^{MHD} \frac{k_B}{\frac{M + m}{2}} T^{MHD} = \rho^{MHD} \frac{k_B}{\bar{m}} T^{MHD}$$

$$T^p \sim T^e \sim T^{MHD}$$

$$n = \frac{\rho^{MHD}}{M + m}$$

$$\bar{m} \equiv \frac{M + m}{2}$$

mean mass
for fully ionized
hydrogen plasma

Equation of state in MHD

$$P^{MHD} = \rho^{MHD} \frac{k_B}{\bar{m}} T^{MHD}$$

Charge conservation equation

$$\left[\begin{array}{l} \frac{e}{M} \times \frac{\partial \rho^p}{\partial t} + \nabla \cdot (\rho^p \mathbf{v}^p) = 0 \quad \dots \text{proton's mass conservation} \\ \rho^p \equiv n_p M \\ \frac{e}{m} \times \frac{\partial \rho^e}{\partial t} + \nabla \cdot (\rho^e \mathbf{v}^e) = 0 \quad \dots \text{electron's mass conservation} \\ \rho^e \equiv n_e m \end{array} \right.$$

subtraction \rightarrow

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad \begin{array}{l} \rho_c \equiv e (n_p - n_e) \\ \mathbf{j} \equiv e (n_p \mathbf{v}^p - n_e \mathbf{v}^e) \end{array}$$

Charge conservation equation
(general version)

In MHD, $n_p \sim n_e \sim n$, so $\rho_c^{MHD} \sim 0$

$$\nabla \cdot \mathbf{j}^{MHD} \sim 0 \quad \mathbf{j}^{MHD} \equiv ne (\mathbf{v}^p - \mathbf{v}^e)$$

Charge conservation equation in MHD

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}^{MHD} \text{ is consistent with this equation.}$$

MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \dots \text{mass conservation}$$

Fluid part

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mathbf{j} \times \mathbf{B} + \mathbf{F} \quad \dots \text{momentum equation}$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} \right) + \nabla \cdot \left(\frac{p}{\gamma - 1} \mathbf{v} \right) = -p \nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{F}_c + \eta j^2 \quad \dots \text{internal energy equation}$$

$$\mathbf{F}_c = -\kappa_c \nabla T$$

$$p = \rho \frac{k_B}{\bar{m}} T \quad \dots \text{equation of state}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta_{diff} \nabla \times \mathbf{B}) \quad \dots \text{induction equation}$$

Electromagnetic part

$$\nabla \cdot \mathbf{B} = 0 \quad \dots \text{magnetic flux conservation (initial condition for magnetic field)}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad \dots \text{Ampere's law}$$

$$\mathbf{E} = \eta \mathbf{j} - \mathbf{v} \times \mathbf{B} \quad \dots \text{Ohm's law (Hall electric field is not included)}$$

We have 9 quantities: $\rho, v_x, v_y, v_z, P, T, B_x, B_y, B_z$

We have 9 equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \dots \text{ for } \rho$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = - \nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \mathbf{F} \quad \dots \text{ for } v_x, v_y, v_z$$

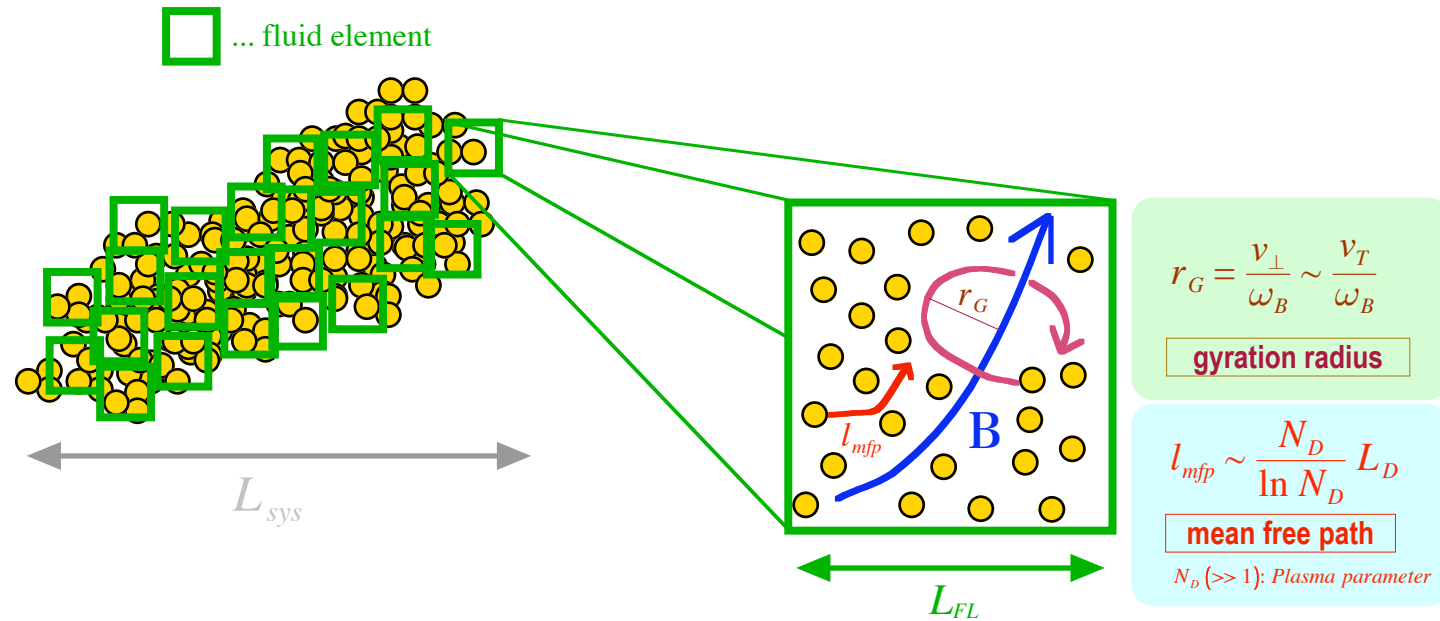
$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} \right) + \nabla \cdot \left(\frac{p}{\gamma - 1} \mathbf{v} \right) = - p \nabla \cdot \mathbf{v} + \nabla \cdot (\kappa_c \nabla T) + \eta j^2 \quad \dots \text{ for } P$$

$$p = \rho \frac{k_B}{m} T \quad \dots \text{ for } T$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta_{diff} \nabla \times \mathbf{B}) \quad \dots \text{ for } B_x, B_y, B_z$$

Summary of MHD

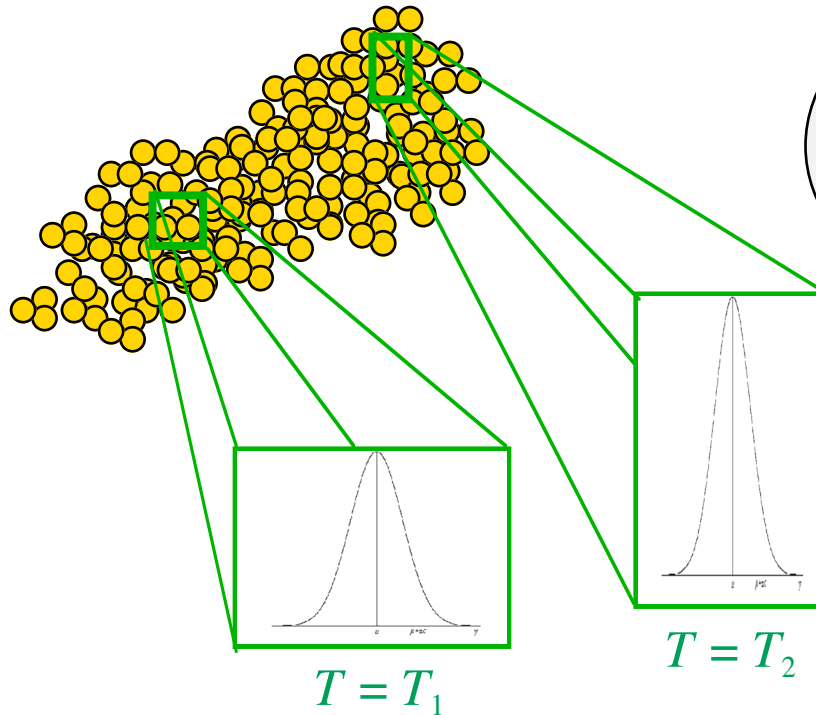
I. Fluid description



When typical length scale of a plasma system is much larger than mean free path and/or gyration radius, fluid element can be defined. => **Fluid approach**

When typical length scale of a plasma system is much smaller than mean free path and gyration radius, fluid element cannot be defined. => **Kinetic approach**

II. Local Thermal Equilibrium (LTE)



Maxwellian distribution

$$f(v_x) = \left(\frac{m}{2\pi k_B T} \right)^{\frac{1}{2}} \exp \left(-\frac{\frac{1}{2} m v_x^2}{k_B T} \right)$$

$$f(v_y) = \left(\frac{m}{2\pi k_B T} \right)^{\frac{1}{2}} \exp \left(-\frac{\frac{1}{2} m v_y^2}{k_B T} \right)$$

$$f(v_z) = \left(\frac{m}{2\pi k_B T} \right)^{\frac{1}{2}} \exp \left(-\frac{\frac{1}{2} m v_z^2}{k_B T} \right)$$

※ in the case of photons

⇒ **Plank distribution**

$$n_{BE}(\nu) = \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}$$

In each fluid element, *random velocity component* of protons & electrons follows **Maxwellian distribution** characterized by a single physical quantity, *temperature*.

When temperature is **different between fluid elements**, it is called **Local Thermal Equilibrium**.

When temperature is **the same in all of them**, it is called **Global Thermal Equilibrium**.

III. Non-relativistic approximation

$$v_0 \ll c$$

IV. Single-fluid model

- Local charge neutrality:

$$n_p \sim n_e \sim n$$

- Density of MHD fluid element is close to proton's density:

$$\rho^{MHD} \equiv n M + n m = n M \left(1 + \frac{m}{M} \right) \approx n M$$

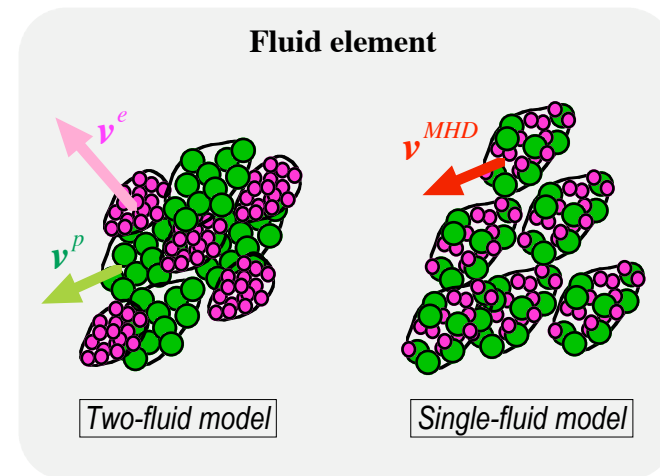
$\frac{1}{1830} \ll 1$

- Velocity of MHD fluid element is close to proton's velocity:

$$\mathbf{v}^{MHD} \equiv \frac{M \mathbf{v}^p + m \mathbf{v}^e}{M + m} \approx \mathbf{v}^p$$

- Velocity of MHD fluid element in B_{\perp} -plane is given by $\mathbf{E}_{conv} \times \mathbf{B}$ drift:

$$\frac{\mathbf{E}_{conv} \times \mathbf{B}}{B^2} = \frac{(-\mathbf{v}^{MHD} \times \mathbf{B}) \times \mathbf{B}}{B^2} = \frac{(-\mathbf{v}_{\perp}^{MHD} \times \mathbf{B}) \times \mathbf{B}}{B^2} = \mathbf{v}_{\perp}^{MHD}$$



V. Diffusion

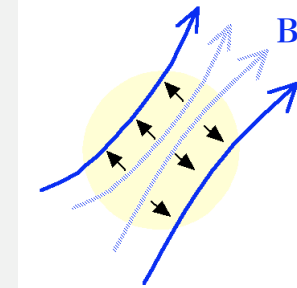
dimension of diffusivity \Rightarrow velocity $\left(\frac{l}{\tau_c}\right) \times$ length $(l) = \frac{l^2}{\tau_c}$

• Magnetic diffusivity

$$\eta_{diff} \equiv \frac{\eta}{\mu_0} = 5.2 \times 10^7 \frac{\ln N_D}{T^{3/2}} \text{ (m}^2/\text{s)} \propto T^{-3/2}$$

$\ln N_D \sim 10$ (Coulomb logarithm)

※ what is diffused is magnetic field ($l = l_{inertial}$).



• Kinematic viscosity

$$\nu = 2.21 \times 10^{-16} \frac{T^{5/2}}{\rho \ln N_D} \text{ (m}^2/\text{s)} \propto T^{5/2} \rho^{-1}$$

※ what is mainly diffused is proton ($l = l_{mfp}^{p-p}$).



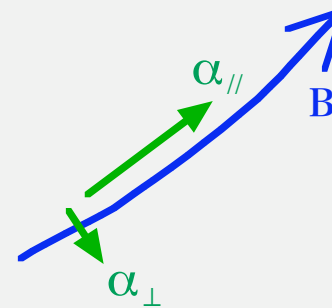
• Thermal diffusivity

$$\alpha \equiv \frac{\kappa_c}{\rho c_p}$$

$$\alpha_{||} = 4.3 \times 10^{-16} \frac{T^{5/2}}{\rho \ln N_D} \text{ (m}^2/\text{s)} \propto T^{5/2} \rho^{-1}$$

$$\frac{\alpha_{\perp}}{\alpha_{||}} = 2.1 \times 10^{-31} \frac{n^2}{T^3 B^2}$$

※ what is mainly diffused is electron ($l = l_{mfp}^{e-e}$)



Roles of Electric Field & Magnetic Field in MHD

Role of Electric Field

It drives evolution of magnetic field.

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}^{MHD}$$

Electric field in MHD => two major components

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}^{MHD} = -\nabla \times \underbrace{(-\mathbf{v} \times \mathbf{B})}_{\text{convective term}} - \nabla \times \underbrace{(\eta_{diff} \nabla \times \mathbf{B})}_{\text{diffusion term}}$$
$$\mathbf{E}_{conv} \equiv -\mathbf{v} \times \mathbf{B} \qquad \mathbf{E}_{resis} \equiv \eta_{diff} \nabla \times \mathbf{B}$$

• **Convective term:** $-\nabla \times \mathbf{E}_{conv} = \nabla \times (\mathbf{v} \times \mathbf{B})$

it drives evolution of magnetic field via **macroscale motion (flow) of fluid elements.**

• **Diffusion term:** $-\nabla \times \mathbf{E}_{resis} = -\nabla \times (\eta_{diff} \nabla \times \mathbf{B})$

it drives evolution of magnetic field via **microscale motion (collision) of particles.**

Convective term: $-\nabla \times \mathbf{E}_{conv} = \nabla \times (\mathbf{v} \times \mathbf{B})$

Convective limit...

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{E}_{conv} + \mathbf{E}_{resis})$$

$$\frac{E_{conv}}{E_{resis}} \sim \frac{v_0 B_0}{\eta_{diff} \frac{B_0}{l_0}} = \frac{v_0 l_0}{\eta_{diff}}$$

$$R_m \equiv \frac{v_0 l_0}{\eta_{diff}} \gg 1$$

we can neglect diffusion term
(convective limit)

Three conditions for *convective limit*:

- I. typical length is very large
- II. typical velocity is very large
- III. magnetic diffusivity is small

e.g. solar coronal loop...

condition I: $l_0 \sim 10^7$ m

condition II: $v_0 \sim 10^5$ m/s

condition III: $\eta_{diff} = 5.2 \times 10^7 \frac{\ln N_D}{T^{3/2}} \sim 0.1$ (m²/s)

$R_m \sim 10^{13} \gg 1$