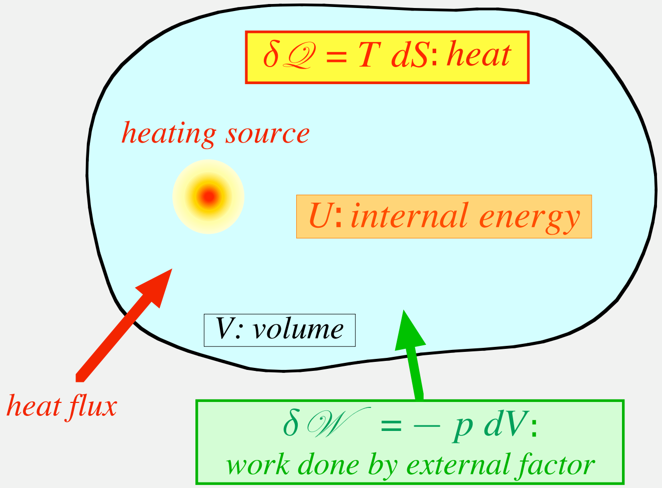
Equation III... internal energy equation

The first law of thermodynamics: $\delta \mathcal{Q} + \delta \mathcal{W} = dU$

$\Rightarrow \; T\,dS - p\,dV = dU$



$\delta \mathcal{Q}$... heat

$\delta \mathcal{W}$ ... work

=> ***process-dependent quantities***

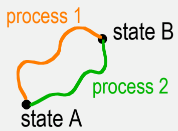
S... entropy

V... volume

p... gas pressure

U... internal energy

=> ***state-dependent quantities***



$\delta \mathcal{Q}_{process\,1} \neq \delta \mathcal{Q}_{process\,2}$

$dV_{A \to B} = V_B - V_A$

uniquely determined from states A & B

Internal energy equation (per unit volume):

$$\frac{1}{V}\,T\,\frac{dS}{dt} - \frac{1}{V}\,p\,\frac{dV}{dt} = \frac{1}{V}\,\frac{dU}{dt}$$

We then rewrite this equation using MHD quantities.

$$\frac{1}{V}\frac{dU}{dt}$$ ... rate of internal energy change

$$U = c_V\, T$$

Internal energy
for perfect gas
(per unit mass)

c_p... *specific heat at constant pressure*

c_v... *specific heat at constant volume*

$$c_p = c_v + \frac{k_B}{m}$$

$$\gamma = \frac{c_p}{c_v} = \frac{c_v + \frac{k_B}{m}}{c_v}$$

ratio of specific heats

$$c_V = \frac{N_f}{2}\frac{k_B}{m}$$

N_f: *number of freedom*

k_B: *Boltzmann constant*

m: *mass of a particle*

$$c_v = \frac{1}{\gamma - 1}\frac{k_B}{m}$$

$$U = c_V\, T = \frac{1}{\gamma - 1}\frac{k_B}{m}\left(\frac{m}{k_B}\frac{p}{\rho}\right) = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Equation of state *(perfect gas)*: $p = \rho\,\frac{k_B}{m}\,T$

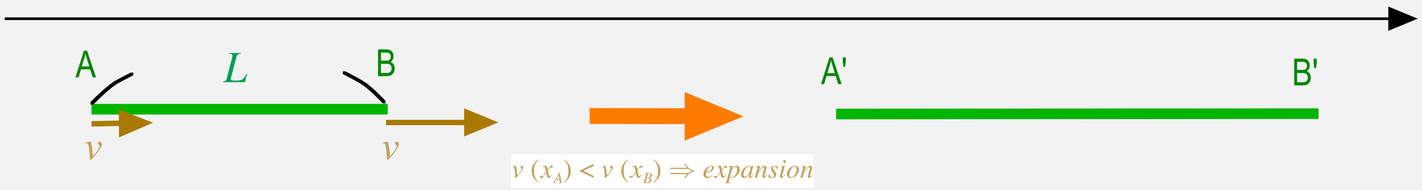
$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)$$ ... per unit mass ($\rho\, V = 1$)

$$\frac{1}{V}\frac{dU}{dt} = \rho\,\frac{d}{dt}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)$$

per unit volume

$$\frac{1}{V}\frac{dV}{dt}$$ ... expansion rate

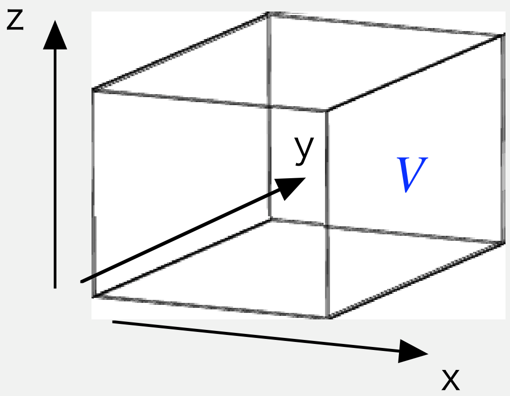
1-dimensional case



$$\text{Expansion rate} \equiv \frac{1}{L}\frac{dL}{dt} \Rightarrow \frac{1}{x_B - x_A}\frac{d}{dt}\left(x_B - x_A\right) \underset{\substack{x_A = x \\ x_B = x + \Delta x}}{\Rightarrow} \frac{1}{\Delta x}\left[v_x(x+\Delta x) - v_x(x)\right] \underset{\Delta x \to 0}{\to} \frac{\partial v_x}{\partial x}$$

3-dimensional case

per unit volume

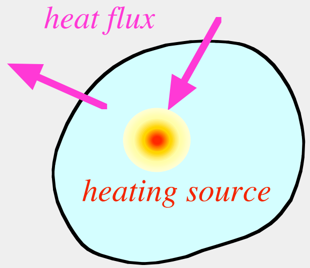


$$\text{Expansion rate} \equiv \frac{1}{V}\frac{dV}{dt} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \equiv \nabla \bullet \boldsymbol{v}$$

$$\frac{1}{V}T\frac{dS}{dt}$$ ... heating rate

heating = **net heat flux** (external origin) + **heating source** (internal origin)

"net flux" => $-\nabla\bullet$



$$\frac{1}{V}T\frac{dS}{dt} = -\nabla\bullet\boldsymbol{F}_c + \eta j^2 + H$$

per unit volume

thermal conduction flux

$$\boldsymbol{F}_c = -\kappa_c \nabla T$$

κ_c: *thermal conductivity*

Joule heating source
(from Ohm's law)

Other heating sources
viscous heating
...

$$\frac{1}{V}T\frac{dS}{dt} = 0 \text{... } \textit{Adiabatic process}$$

There is neither net heat flux nor heating source. Entropy $dS = \frac{\delta Q}{T}$ is conserved.

$$\frac{1}{V} T \frac{dS}{dt} - \frac{1}{V} p \frac{dV}{dt} = \frac{1}{V} \frac{dU}{dt}$$

per unit volume

$$\frac{1}{V}\frac{dU}{dt} \rightarrow \rho \frac{d}{dt}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)$$

$$\frac{1}{V} p \frac{dV}{dt} \rightarrow p \nabla \bullet \boldsymbol{v}$$

$$\frac{1}{V} T \frac{dS}{dt} \rightarrow -\nabla \bullet \boldsymbol{F}_c + \eta j^2 + H$$

Internal energy eq. in MHD

$$\rho \frac{d}{dt}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right) = - p \nabla \bullet \boldsymbol{v} - \nabla \bullet \boldsymbol{F}_c + \eta j^2 + H \quad + \text{radiation (flux)}$$

- $- p \nabla \bullet \boldsymbol{v}$: **compression** (source; reversible)
- $- \nabla \bullet \boldsymbol{F}_c$: **conduction** (flux)
- $\eta j^2 + H$: **heating** (source; irreversible)

$$= \rho \frac{\partial}{\partial t}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right) + \rho \boldsymbol{v} \bullet \nabla \left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right) = \frac{\partial}{\partial t}\left(\frac{p}{\gamma - 1}\right) + \nabla \bullet \left(\frac{p}{\gamma - 1}\boldsymbol{v}\right)$$

- $\nabla \bullet \left(\frac{p}{\gamma - 1}\boldsymbol{v}\right)$: **convection** (flux)