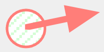
Equation II... momentum equation

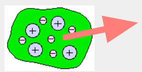
Equation of motion for a particle: $M\dfrac{d\boldsymbol{v}}{dt} = \boldsymbol{F}$ **(mass x acceleration = force)**



$\boldsymbol{F}_{em} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$

$\boldsymbol{F}_g = M\,\boldsymbol{g}$

Momentum equation for a fluid element: $\rho\dfrac{d\boldsymbol{v}}{dt} = \boldsymbol{f}$



mass density (mass per unit volume)

acceleration

volume force (force per unit volume)

- Acceleration: $\dfrac{d\boldsymbol{v}}{dt}$ *... Lagrangian derivative* $\Rightarrow \dfrac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \bullet \nabla\, \boldsymbol{v}$ *... Eulerian derivative*
- Volume force

  gas pressure gradient force (*collision-related*): $\boldsymbol{f}_p = -\nabla\, p$

  Coulomb force + Lorentz force (*collision-unrelated; interaction with field*): $\boldsymbol{f}_{em} = n\,q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)$

  gravitational force (*collision-unrelated; interaction with field*): $\boldsymbol{f}_g = n\,M\,\boldsymbol{g}$

  viscous force (*collision-related*): $\boldsymbol{f}_v = \rho\,\nu\left[\nabla^2 \boldsymbol{v} + \dfrac{1}{3}\nabla\left(\nabla \bullet \boldsymbol{v}\right)\right]$, ν*: viscosity*

Lagrangian derivative vs. Eulerian derivative

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\boldsymbol{v} \bullet \nabla)$$

$$d\,A(x, y, z, t) = \frac{\partial A}{\partial x} d\,x + \frac{\partial A}{\partial y} d\,y + \frac{\partial A}{\partial z} d\,z + \frac{\partial A}{\partial t} d\,t$$

total differential

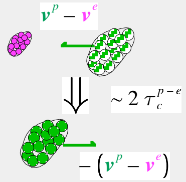
$$\begin{aligned}
\therefore \frac{d\,A}{d\,t} &= \frac{\partial A}{\partial x}\frac{d\,x}{d\,t} + \frac{\partial A}{\partial y}\frac{d\,y}{d\,t} + \frac{\partial A}{\partial z}\frac{d\,z}{d\,t} + \frac{\partial A}{\partial t}\frac{d\,t}{d\,t} \\
&= \frac{\partial A}{\partial x} v_x + \frac{\partial A}{\partial y} v_y + \frac{\partial A}{\partial z} v_z + \frac{\partial A}{\partial t} \\
&= \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) A + \frac{\partial A}{\partial t} \\
&= \frac{\partial A}{\partial t} + (\boldsymbol{v} \bullet \nabla)\, A
\end{aligned}$$

Momentum equation in MHD (simplified version... nonlinear term $(\boldsymbol{v} \bullet \nabla)\boldsymbol{v}$ is neglected)

Proton's momentum equation (nonlinear term $(\boldsymbol{v}^p \bullet \nabla)\boldsymbol{v}^p$ is neglected):

$$n\,M\,\frac{\partial \boldsymbol{v}^p}{\partial t} = -\nabla P^p + n\,e\left(\boldsymbol{E} + \boldsymbol{v}^p \times \boldsymbol{B}\right) + \boldsymbol{f}_c^{\,p-e}$$

$$\boldsymbol{f}_c^{\,p-e} \approx -n\,M\,\frac{\boldsymbol{v}^p - \boldsymbol{v}^e}{\tau_c^{p-e}}$$



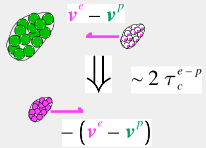
macroscale electric field
(e.g. $\boldsymbol{E}_{convective} = -\boldsymbol{v} \times \boldsymbol{B}$)

collision force (*microscale electric field* characterized by L_D)
= change in *momentum* (*relative velocity*) per unit time

Electron's momentum equation (nonlinear term $(\boldsymbol{v}^e \bullet \nabla)\boldsymbol{v}^e$ is neglected):

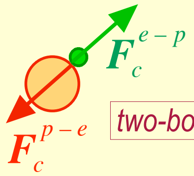
$$n\,m\,\frac{\partial \boldsymbol{v}^e}{\partial t} = -\nabla P^e - n\,e\left(\boldsymbol{E} + \boldsymbol{v}^e \times \boldsymbol{B}\right) + \boldsymbol{f}_c^{\,e-p}$$

$$\boldsymbol{f}_c^{\,e-p} \approx -n\,m\,\frac{\boldsymbol{v}^e - \boldsymbol{v}^p}{\tau_c^{e-p}}$$



Conservation of total momentum via collision

$$\boldsymbol{f}_c^{\,e-p} + \boldsymbol{f}_c^{\,p-e} = \boldsymbol{0} \quad \left(\Rightarrow \tau_c^{e-p} = \frac{m}{M}\,\tau_c^{p-e}\right)$$



two-body model

Add both equations...

$$n\,\frac{\partial}{\partial t}\left(M\,\boldsymbol{v}^p + m\,\boldsymbol{v}^e\right) = n\,e\left(\boldsymbol{v}^p - \boldsymbol{v}^e\right) \times \boldsymbol{B} - \nabla\left(P^p + P^e\right)$$

$$\rho^{MHD}\,\frac{\partial}{\partial t}\,\boldsymbol{v}^{MHD} = \boldsymbol{j}^{MHD} \times \boldsymbol{B} - \nabla P^{MHD}$$

$P^{MHD} \equiv P^p + P^e$: *total gas pressure*

$\boldsymbol{j}^{MHD} \equiv n\,e\left(\boldsymbol{v}^p - \boldsymbol{v}^e\right)$: *current density*

Momentum eq. in MHD
(simplified version)

Nonlinear term of flow velocity: $(\boldsymbol{v} \bullet \nabla)\boldsymbol{v}$

$$\left(\rho^{p}\, v^{p}{}_{k}\right) \frac{\partial}{\partial x_k} v^{p}{}_{i} + \left(\rho^{e}\, v^{e}{}_{k}\right) \frac{\partial}{\partial x_k} v^{e}{}_{i} = n\, M \left(v^{p}{}_{k} \frac{\partial}{\partial x_k} v^{p}{}_{i} + \frac{m}{M} v^{e}{}_{k} \frac{\partial}{\partial x_k} v^{e}{}_{i} \right) = n\, M \left[v^{p}{}_{k} \frac{\partial}{\partial x_k} v^{p}{}_{i} \left(\frac{m}{M}\right)^{0} + O\left(\left(\frac{m}{M}\right)^{1}\right)\right]$$

1st-order term of $\frac{m}{M}$

$$\left(\rho^{MHD}\, v^{MHD}{}_{k}\right) \frac{\partial}{\partial x_k} v^{MHD}{}_{i} = n \left(M + m\right) \frac{M\, v^{p}{}_{k} + m\, v^{e}{}_{k}}{M + m} \frac{\partial}{\partial x_k} \left(\frac{M\, v^{p}{}_{i} + m\, v^{e}{}_{i}}{M + m}\right)$$

$$= n\, M \left(v^{p}{}_{k} + \frac{m}{M} v^{e}{}_{k}\right) \frac{\partial}{\partial x_k} \left(\frac{v^{p}{}_{i} + \frac{m}{M} v^{e}{}_{i}}{1 + \frac{m}{M}}\right) = n\, M\, v^{p}{}_{k} \frac{\partial}{\partial x_k} v^{p}{}_{i} \left(\frac{m}{M}\right)^{0} + O\left(\left(\frac{m}{M}\right)^{1}, \left(\frac{m}{M}\right)^{2}, \ldots\right)$$

1st-order and even higher-order terms of $\frac{m}{M}$

comparison

$$\left(\rho^{p}\, v^{p}{}_{k}\right) \frac{\partial}{\partial x_k} v^{p}{}_{i} + \left(\rho^{e}\, v^{e}{}_{k}\right) \frac{\partial}{\partial x_k} v^{e}{}_{i} \approx \left(\rho^{MHD}\, v^{MHD}{}_{k}\right) \frac{\partial}{\partial x_k} v^{MHD}{}_{i}$$

based on the 0th-order approximation of $\frac{m}{M}$

$$\rho^{MHD}\left[\frac{\partial}{\partial t}\boldsymbol{v}^{MHD} + \left(\boldsymbol{v}^{MHD} \bullet \nabla\right)\boldsymbol{v}^{MHD}\right] = \boldsymbol{j}^{MHD} \times \boldsymbol{B} - \nabla P^{MHD}$$

Momentum eq. in MHD