MHD induction equation

... represents **electromagnetic part of MHD**

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ Faraday's law

$$\boldsymbol{E} = \eta\, \boldsymbol{j} - \boldsymbol{v} \times \boldsymbol{B}$$ Ohm's law

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left(-\boldsymbol{v} \times \boldsymbol{B} + \eta\, \boldsymbol{j}\right)$$

$$\nabla \times \boldsymbol{B} = \mu_0\, \boldsymbol{j}^{MHD}$$ Ampere's law

eliminate $\boldsymbol{j}$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left(\boldsymbol{v} \times \boldsymbol{B} - \eta_{diff}\, \nabla \times \boldsymbol{B}\right)$$

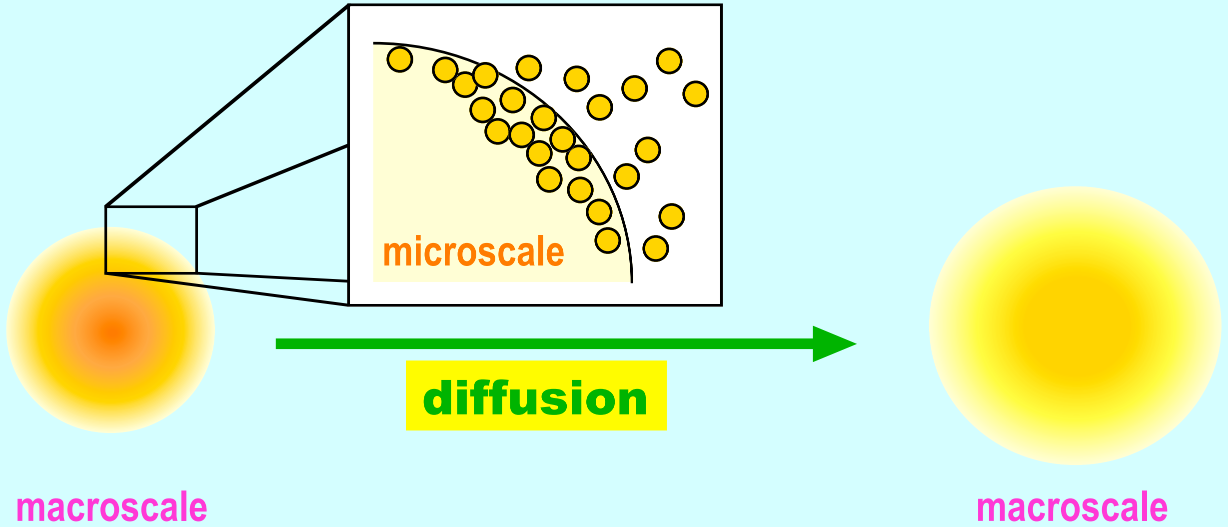
$\eta_{diff} \equiv \frac{\eta}{\mu_0}$ ... *magnetic diffusivity*

MHD induction equation

... describes evolution of magnetic field using plasma quantities

What is diffusion?

A **microscale process** (<= kinetic approach) caused by **collision of particles**, which makes a **macroscale distribution** (<= fluid approach) **smooth**.



Diffusion is characterized by microscale evolution:

=> mean free path/inertial length l & collision time τ_c

dimension of diffusivity (m²/s)... **velocity** (m/s) x **length** (m)

$$v\, l = \frac{l}{\tau_c}\, l = \frac{l^2}{\tau_c}$$

Magnetic diffusivity

Magnetic diffusivity

$\ln N_D$ (Coulomb logarithm) ~ 10

$$\eta_{diff} \equiv \frac{\eta}{\mu_0} = 5.2 \times 10^7 \frac{\ln N_D}{T^{3/2}} \ (\mathrm{m^2/s}) \propto T^{-3/2}$$

inertial length^2 / collision time

※ What is diffused is magnetic field against ions and electrons.

Magnetic diffusivity is **lower** in a **hotter** region.

Electric energy density vs. Magnetic energy density

Electric energy density... $\mathcal{E}_E = \frac{\varepsilon_0}{2} E^2$

Magnetic energy density... $\mathcal{E}_M = \frac{B^2}{2\mu_0}$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \longrightarrow \frac{E_0}{l_0} \sim \frac{B_0}{t_0}$$

The ratio is

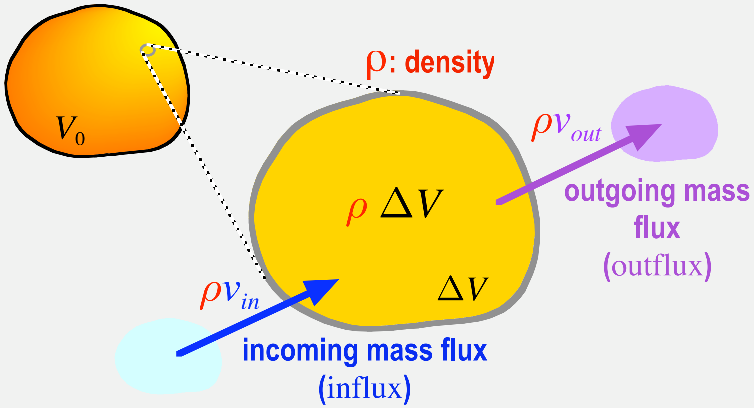
$$\mathcal{E}_E / \mathcal{E}_M = \frac{\varepsilon_0 \mu_0 E^2}{B^2} \approx \frac{\left(\frac{l_0}{t_0} B_0\right)^2}{c^2 B_0^2} = \frac{v_0^2}{c^2} << 1$$

→ Electric energy density is negligible when $v_0 << c$.

Fluid part
- mass conservation
- momentum equation
- energy equation
- equation of state
- Ohm's law
- charge conservation

(**Fields** => **Plasma**)

Equation I... mass conservation



$$\frac{\Delta}{\Delta t}(\rho\ \Delta V) \Leftarrow net\ flux \equiv influx - outflux$$

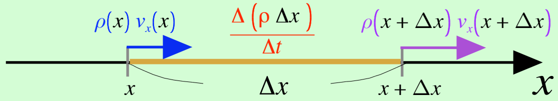
1D case

x and t are independent

$$\Delta x\,\frac{\Delta\,(\rho\ \Delta x)}{\Delta t} = \rho(x)\ v_x(x) - \rho(x+\Delta x)\ v_x(x+\Delta x)$$

$$\Rightarrow \frac{\Delta\ \rho}{\Delta t} = \frac{\rho(x)\ v_x(x) - \rho(x+\Delta x)\ v_x(x+\Delta x)}{\Delta x}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho\ v_x)$$



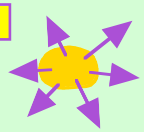
3D case

$$\frac{\partial \rho}{\partial t} = -\nabla \bullet (\rho\ \boldsymbol{v}) \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \bullet (\rho\ \boldsymbol{v}) = 0$$

$$\nabla \bullet (\rho\ \boldsymbol{v}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \bullet (\rho\ v_x, \rho\ v_y, \rho\ v_z)$$
$$= \frac{\partial\,(\rho\ v_x)}{\partial x} + \frac{\partial\,(\rho\ v_y)}{\partial y} + \frac{\partial\,(\rho\ v_z)}{\partial z}$$

$\nabla \bullet (\rho\ \boldsymbol{v})$... $-$ net flux; when this is positive, density decreases with time.

divergence



$$\nabla \bullet (\rho\ \boldsymbol{v}) > 0 \ \Rightarrow\ \frac{\partial \rho}{\partial t} < 0$$

Mass conservation equation in MHD

$$\frac{\partial \rho^p}{\partial t} + \nabla \bullet \left(\rho^p \, \boldsymbol{v}^p\right) = 0$$ ... proton's mass conservation $\rho^p = n\,M$

$$\frac{\partial \rho^e}{\partial t} + \nabla \bullet \left(\rho^e \, \boldsymbol{v}^e\right) = 0$$ ... electron's mass conservation $\rho^e = n\,m$

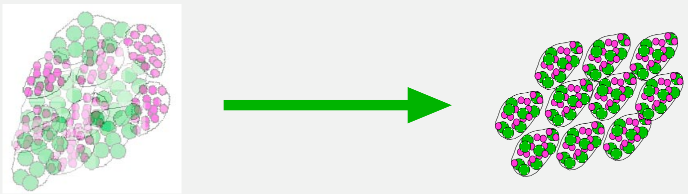
summation

$$\frac{\partial \rho^{MHD}}{\partial t} + \nabla \bullet \left(\rho^{MHD} \, \boldsymbol{v}^{MHD}\right) = 0$$

Mass conservation eq. in MHD

$\rho^{MHD} \equiv n\left(M + m\right)$: *total density*

$\boldsymbol{v}^{MHD} \equiv \dfrac{M\,\boldsymbol{v}^p + m\,\boldsymbol{v}^e}{M + m}$: *mass-average velocity*



two-species fluid elements (proton + electron)

single-species fluid elements (MHD)