

Momentum conservation (1D, steady condition):

$$\frac{\partial}{\partial t} (\rho v_x) + \frac{\partial}{\partial x} (\rho v_x v_x + p) = \frac{\partial \pi_{xx}}{\partial x} \rightarrow \frac{d}{dx} (\rho v_x v_x + p) = \frac{d \pi_{xx}}{dx}$$

$$\int_{x_2}^{x_1} \frac{d}{dx} (\rho v_x v_x + p) dx = [\rho v_x^2 + p]_{x_2}^{x_1} = \rho_1 v_{x1}^2 + p_1 - (\rho_2 v_{x2}^2 + p_2) = \frac{\pi_{xx}(x_1)}{=0} - \frac{\pi_{xx}(x_2)}{=0} \quad \pi_{xx} = \frac{4}{3} \mu \frac{\partial v_x}{\partial x}$$

$$\rightarrow \rho_1 v_{x1}^2 + p_1 = \rho_2 v_{x2}^2 + p_2$$

because v_x is uniform just outside the shock layer

This means that we do not neglect diffusion even though the final relation apparently has only ideal terms and no diffusion term. This is partially demonstrated by the entropy relation discussed later.

Energy conservation (1D, steady condition):

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v_x^2 + \frac{1}{\gamma - 1} p \right) + \frac{\partial}{\partial x} \left[\left(\frac{1}{2} \rho v_x^2 + \frac{1}{\gamma - 1} p + p \right) v_x \right] = q$$

$$\rightarrow \frac{d}{dx} \left[\left(\frac{1}{2} \rho v_x^2 + \frac{1}{\gamma - 1} p + p \right) v_x \right] = q$$

$$\int_{x_2}^{x_1} \frac{d}{dx} \left[\left(\frac{1}{2} \rho v_x^2 + \frac{1}{\gamma - 1} p + p \right) v_x \right] dx = \left[\left(\frac{1}{2} \rho v_x^2 + \frac{\gamma}{\gamma - 1} p \right) v_x \right]_{x_2}^{x_1}$$

$$= \left(\frac{1}{2} \rho_1 v_{x1}^2 + \frac{\gamma}{\gamma - 1} p_1 \right) v_{x1} - \left(\frac{1}{2} \rho_2 v_{x2}^2 + \frac{\gamma}{\gamma - 1} p_2 \right) v_{x2} = 0$$

$$\rightarrow \left(\frac{1}{2} \rho_1 v_{x1}^2 + \frac{\gamma}{\gamma - 1} p_1 \right) v_{x1} = \left(\frac{1}{2} \rho_2 v_{x2}^2 + \frac{\gamma}{\gamma - 1} p_2 \right) v_{x2}$$

$$\ast \int_{x_2}^{x_1} q dx = 0$$

$$q = \frac{\partial}{\partial x} (\pi_{xx} v_x - \mathcal{E}_x)$$

$$\mathcal{E}_x = -\kappa_e \frac{\partial T}{\partial x}$$

because v_x & T are uniform just outside the shock layer

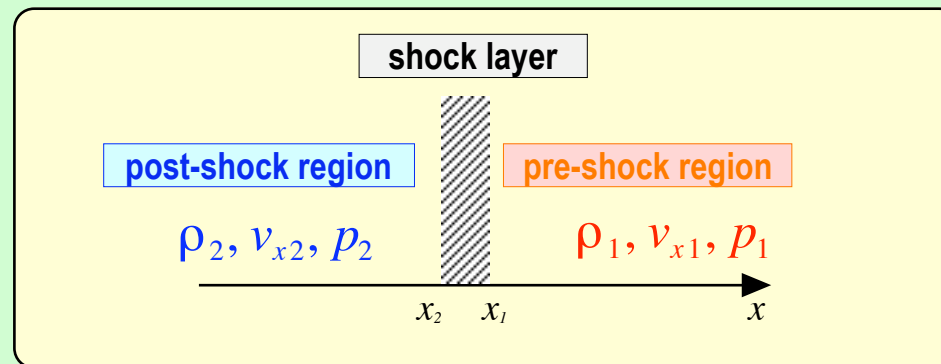
Relations between pre-shock and post-shock regions

=> **Rankine-Hugoniot relations**

$$\rho_1 v_{x1} = \rho_2 v_{x2}$$

$$\rho_1 v_{x1}^2 + p_1 = \rho_2 v_{x2}^2 + p_2$$

$$\left(\frac{1}{2} \rho_1 v_{x1}^2 + \frac{\gamma}{\gamma - 1} p_1 \right) v_{x1} = \left(\frac{1}{2} \rho_2 v_{x2}^2 + \frac{\gamma}{\gamma - 1} p_2 \right) v_{x2}$$



From these three equations, we derive **ratios of three physical quantities** $\frac{\rho_2}{\rho_1}, \frac{v_{x2}}{v_{x1}}, \frac{p_2}{p_1}$ across the **shock layer**.

From **R-H relations**, we obtained the following equation for $X = \frac{\rho_2}{\rho_1}$

$$\left(\frac{\gamma - 1}{2} M_1^2 + 1 \right) X^2 - \left(\gamma M_1^2 + 1 \right) X + \frac{\gamma + 1}{2} M_1^2 = 0$$

$$M_1 \equiv \frac{v_{x1}}{\sqrt{\gamma p_1 / \rho_1}} = \frac{v_{x1}}{c_{s1}} \text{ is Mach number in the pre-shock region.}$$

There are two mathematical solutions, namely,

$$X = \frac{(\gamma + 1) M_1^2}{(\gamma - 1) M_1^2 + 2} \quad \text{or} \quad X = 1$$

The **former** gives a **non-trivial solution** of the **shock wave**.

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_1^2}{2 + (\gamma - 1) M_1^2}$$

$$\frac{v_{x2}}{v_{x1}} = \frac{2 + (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2}$$

$$\frac{p_2}{p_1} = \frac{2 \gamma M_1^2 - (\gamma - 1)}{\gamma + 1}$$

Properties of HD shock wave

Density ratio *has* **the upper limit:**

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_1^2}{2 + (\gamma - 1) M_1^2}$$

... **upper limit:** $\frac{\gamma + 1}{\gamma - 1}$ *when* $M_1 \rightarrow \infty$

Flow velocity ratio *has* **the lower limit:**

$$\frac{v_{x2}}{v_{x1}} = \frac{2 + (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2}$$

... **lower limit:** $\frac{\gamma - 1}{\gamma + 1}$ *when* $M_1 \rightarrow \infty$

Pressure (and temperature) ratio *has* **no upper limit:**

$$\frac{p_2}{p_1} = \frac{2 \gamma M_1^2 - (\gamma - 1)}{\gamma + 1}$$

... **no upper limit** *when* $M_1 \rightarrow \infty$