

Equation of state (fully ionized case)

$$P^p = n k_B T^p \quad \dots \text{proton's equation of state}$$

$$P^e = n k_B T^e \quad \dots \text{electron's equation of state}$$

summation

$$P^{MHD} = 2 n k_B T^{MHD} = 2 \frac{\rho^{MHD}}{M + m} k_B T^{MHD} = \rho^{MHD} \frac{k_B}{\frac{M + m}{2}} T^{MHD} = \rho^{MHD} \frac{k_B}{\bar{m}} T^{MHD}$$

$$T^p \sim T^e \sim T^{MHD}$$

$$n = \frac{\rho^{MHD}}{M + m}$$

$$\bar{m} \equiv \frac{M + m}{2}$$

mean mass
for fully ionized
hydrogen plasma

$$P^{MHD} = \rho^{MHD} \frac{k_B}{\bar{m}} T^{MHD}$$

Equation of state in MHD

Charge conservation equation

$$\begin{array}{l}
 \left[\begin{array}{l} \frac{e}{M} \times \frac{\partial \rho^p}{\partial t} + \nabla \cdot (\rho^p \mathbf{v}^p) = 0 \quad \dots \text{proton's mass conservation} \\ \frac{e}{m} \times \frac{\partial \rho^e}{\partial t} + \nabla \cdot (\rho^e \mathbf{v}^e) = 0 \quad \dots \text{electron's mass conservation} \end{array} \right. \\
 \left. \begin{array}{l} \rho^p \equiv n_p M \\ \rho^e \equiv n_e m \end{array} \right] \\
 \text{subtraction} \rightarrow \frac{\partial \rho_c}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad \begin{array}{l} \rho_c \equiv e (n_p - n_e) \\ \mathbf{j} \equiv e (n_p \mathbf{v}^p - n_e \mathbf{v}^e) \end{array}
 \end{array}$$

Charge conservation equation
(general version)

In MHD, $n_p \sim n_e \sim n$, so $\rho_c^{MHD} \sim 0$

$$\nabla \cdot \mathbf{j}^{MHD} = 0 \quad \mathbf{j}^{MHD} \equiv ne (\mathbf{v}^p - \mathbf{v}^e)$$

Charge conservation equation in MHD

Ampere's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}^{MHD}$ is consistent with this equation.

MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \dots \text{mass conservation}$$

Fluid part

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = - \nabla p + \mathbf{j} \times \mathbf{B} + \mathbf{F} \quad \dots \text{momentum equation}$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} \right) + \nabla \cdot \left(\frac{p}{\gamma - 1} \mathbf{v} \right) = - p \nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{F}_c + \eta j^2 \quad \dots \text{internal energy equation}$$

$$\mathbf{F}_c = - \kappa_c \nabla T$$

$$p = \rho \frac{k_B}{\bar{m}} T \quad \dots \text{equation of state}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta_{diff} \nabla \times \mathbf{B}) \quad \dots \text{induction equation}$$

Electromagnetic part

$$\nabla \cdot \mathbf{B} = 0 \quad \dots \text{magnetic flux conservation (initial condition for magnetic field)}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad \dots \text{Ampere's law}$$

$$\mathbf{E} = \eta \mathbf{j} - \mathbf{v} \times \mathbf{B} \quad \dots \text{Ohm's law (Hall electric field is not included)}$$

We have 9 quantities: $\rho, v_x, v_y, v_z, P, T, B_x, B_y, B_z$

We have 9 equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \dots \text{for } \rho$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = - \nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \mathbf{F} \quad \dots \text{for } v_x, v_y, v_z$$

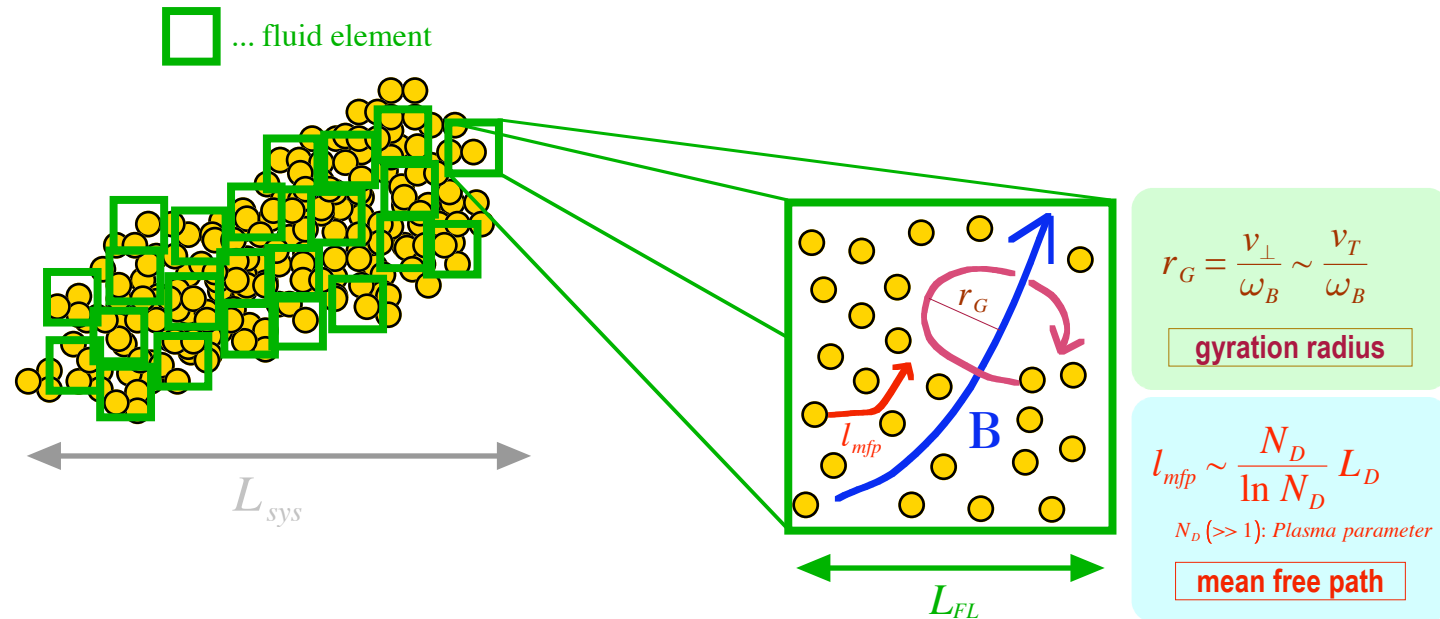
$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} \right) + \nabla \cdot \left(\frac{p}{\gamma - 1} \mathbf{v} \right) = - p \nabla \cdot \mathbf{v} + \nabla \cdot (\kappa_c \nabla T) + \eta j^2 \quad \dots \text{for } P$$

$$p = \rho \frac{k_B}{m} T \quad \dots \text{for } T$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta_{diff} \nabla \times \mathbf{B}) \quad \dots \text{for } B_x, B_y, B_z$$

Summary of MHD

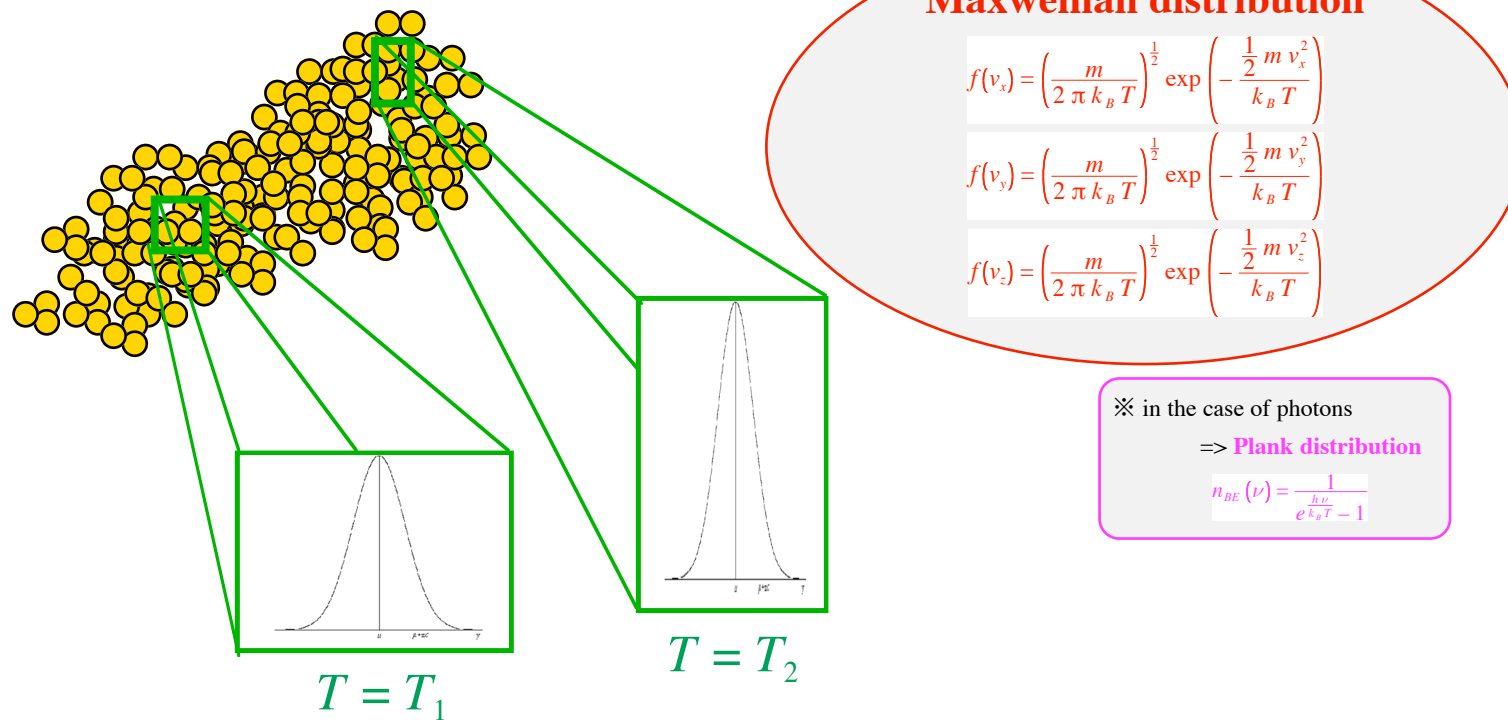
I. Fluid description



When typical length scale of a plasma system is *much larger than* mean free path and/or gyration radius, fluid element can be defined. => **Fluid approach**

When typical length scale of a plasma system is *much smaller than* mean free path and gyration radius, fluid element cannot be defined. => **Kinetic approach**

II. Local Thermal Equilibrium (LTE)



In each **fluid element**, **random velocity component** of protons & electrons follows **Maxwellian distribution** characterized by a single physical quantity, **temperature**.

When temperature is **different between** fluid elements, it is called **Local Thermal Equilibrium**.

When temperature is **the same for all of them**, it is called **Global Thermal Equilibrium**.

III. Non-relativistic approximation

$$v_0 \ll c$$

IV. Single-fluid model

- Local charge neutrality:

$$n_p \sim n_e = n$$

- Density of MHD fluid element is close to proton's density:

$$\rho^{MHD} \equiv n M + n m = n M \left(1 + \frac{m}{M} \right) \approx n M$$

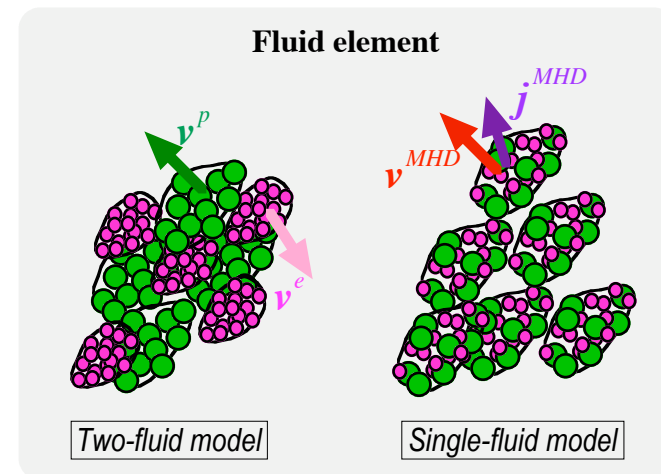
$\frac{1}{1836} \ll 1$

- Velocity of MHD fluid element is close to proton's average velocity:

$$\mathbf{v}^{MHD} \equiv \frac{M \mathbf{v}^p + m \mathbf{v}^e}{M + m} \approx \mathbf{v}^p$$

- Velocity of MHD fluid element in $B_{\perp}$ -plane is given by $\mathbf{E}_{conv} \times \mathbf{B}$ drift:

$$\frac{\mathbf{E}_{conv} \times \mathbf{B}}{B^2} = \frac{(-\mathbf{v}^{MHD} \times \mathbf{B}) \times \mathbf{B}}{B^2} = \frac{(-\mathbf{v}_{\perp}^{MHD} \times \mathbf{B}) \times \mathbf{B}}{B^2} = \mathbf{v}_{\perp}^{MHD}$$



V. Diffusion

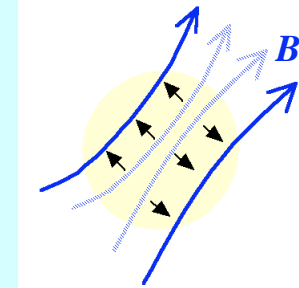
dimension of diffusivity $\Rightarrow$ velocity $\left(\frac{l}{\tau_c}\right) \times$ length $(l) = \frac{l^2}{\tau_c}$

• Magnetic diffusivity

$$\eta_{diff} \equiv \frac{\eta}{\mu_0} = 5.2 \times 10^7 \frac{\ln N_D}{T^{3/2}} \text{ (m}^2/\text{s)} \propto T^{-3/2}$$

$\ln N_D \sim 10$ (Coulomb logarithm)

※ What diffuses is magnetic field ($l = l_{inertial}$).



• Kinematic viscosity

$$\nu = 2.21 \times 10^{-16} \frac{T^{5/2}}{\rho \ln N_D} \text{ (m}^2/\text{s)} \propto T^{5/2} \rho^{-1}$$

※ What mainly diffuses is proton ($l = l_{mfp}^{p-p}$).



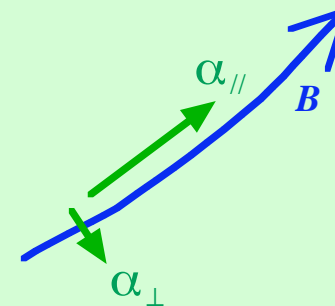
• Thermal diffusivity

$$\alpha \equiv \frac{\kappa_C}{\rho c_p}$$

$$\alpha_{||} = 4.3 \times 10^{-16} \frac{T^{5/2}}{\rho \ln N_D} \text{ (m}^2/\text{s)} \propto T^{5/2} \rho^{-1}$$

$$\frac{\alpha_{\perp}}{\alpha_{||}} = 2.1 \times 10^{-31} \frac{n^2}{T^3 B^2}$$

※ What mainly diffuses is electron ($l = l_{mfp}^{e-e}$).

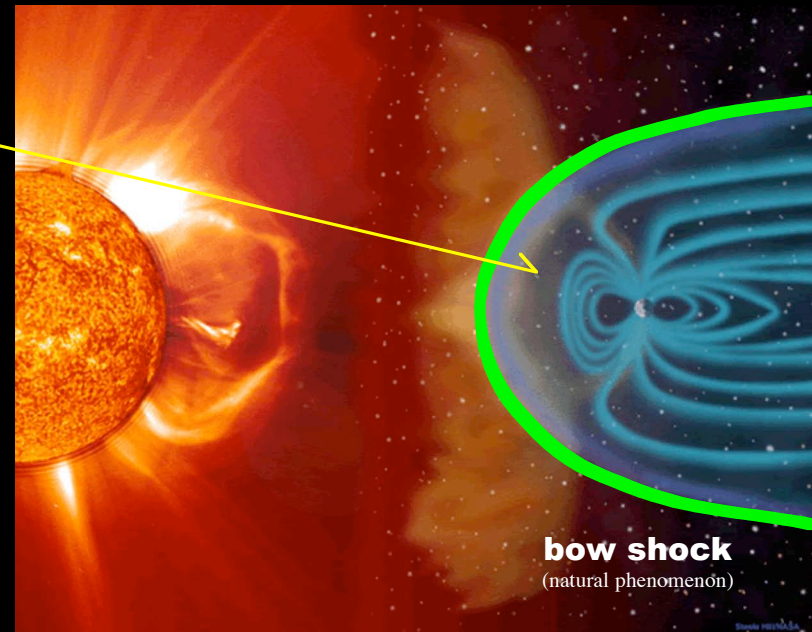


Shock wave

Examples of **shock wave**



Behind a **shock wave** (post-shock region),
flow speed is lower than wave propagation speed
density, pressure, and temperature increase



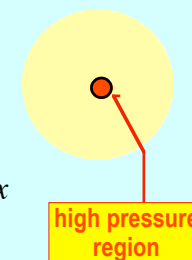
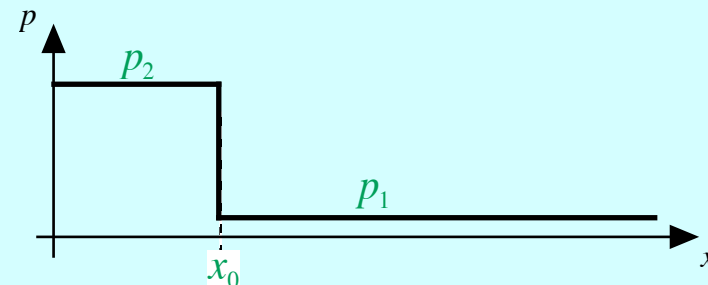
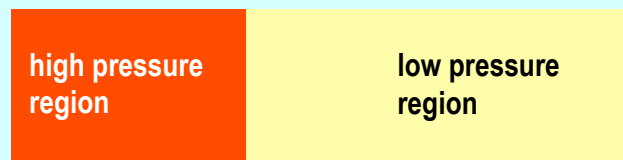
Generating mechanisms of a shock wave

1) *Thermal energy is deposited:* **shock tube, blast wave**

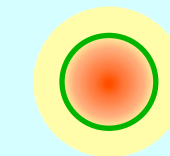
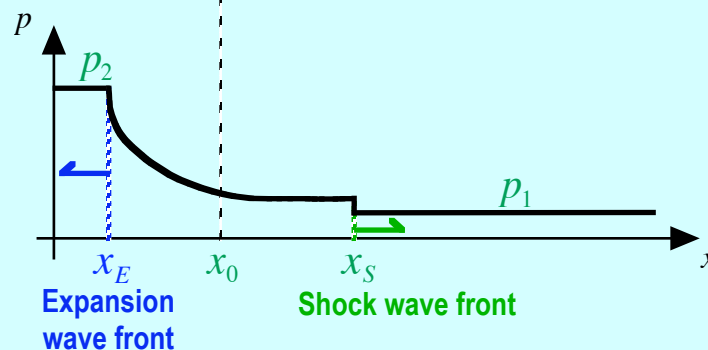
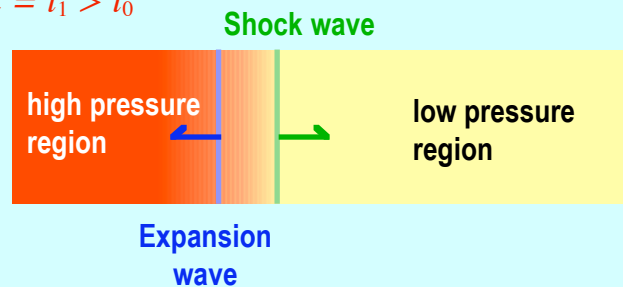
associated with a **highly dynamic** event



$t = t_0$



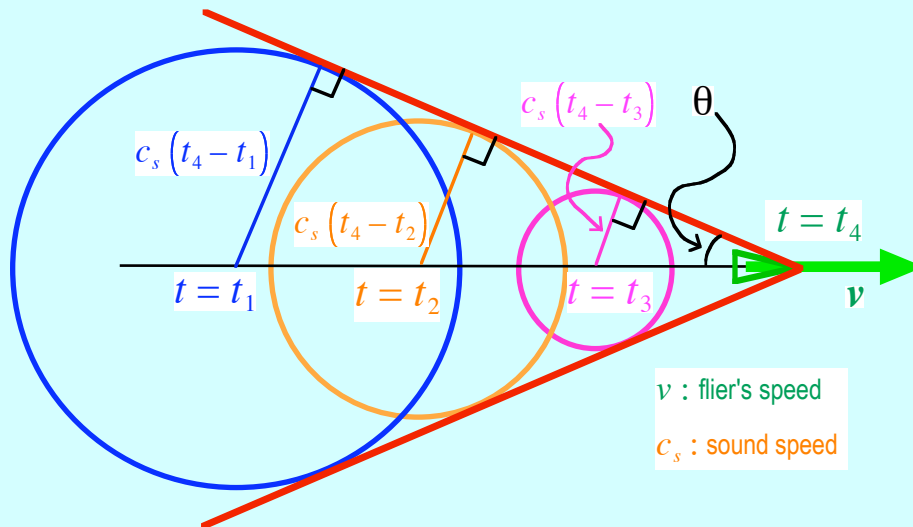
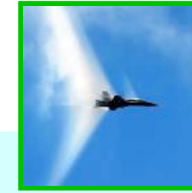
$t = t_1 > t_0$



Toward the low-pressure region... shock wave propagates
Toward the high-pressure region... expansion wave propagates

2) Supersonic flier exists: **supersonic jet plane**

associated with a **highly dynamic** event



A flier (green triangle) travels at a **supersonic speed** ($v > c_s$).

A **sound wave** generated at t_1 , t_2 and t_3 forms a **spherical wave front** (blue, orange, pink).

The **envelop** of these spherical wave fronts forms a **cone-shaped shock wave front** (red).

The **inclination angle** θ is given by

$$v \sin \theta = c_s \quad \left(\Leftrightarrow \frac{v}{c_s} \equiv M = \frac{1}{\sin \theta} > 1 \right)$$

When the **Mach number M** is **much larger than 1**, the shock wave front forms a **sharp cone** (small θ).



When the Mach number M is **not much larger than 1**, the shock wave front is **detached from** the flier and forms an **arc-shaped bow shock** in front of the flier.

