

Mass conservation equation in MHD

$$\left[\begin{array}{l} \frac{\partial \rho^p}{\partial t} + \nabla \cdot (\rho^p \mathbf{v}^p) = 0 \quad \dots \text{proton's mass conservation } \rho^p = n M \\ \frac{\partial \rho^e}{\partial t} + \nabla \cdot (\rho^e \mathbf{v}^e) = 0 \quad \dots \text{electron's mass conservation } \rho^e = n m \end{array} \right.$$

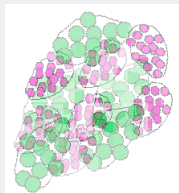
summation

$$\frac{\partial \rho^{MHD}}{\partial t} + \nabla \cdot (\rho^{MHD} \mathbf{v}^{MHD}) = 0$$

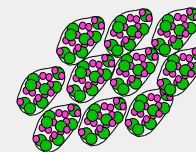
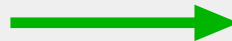
Mass conservation eq. in MHD

$$\rho^{MHD} \equiv n (M + m): \text{total density}$$

$$\mathbf{v}^{MHD} \equiv \frac{M \mathbf{v}^p + m \mathbf{v}^e}{M + m}: \text{mass-average velocity}$$



two-species fluid elements
(proton + electron)



single-species fluid elements
(MHD)

Equation II... **momentum equation**

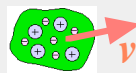
Equation of motion for a particle: $M \frac{d\mathbf{v}}{dt} = \mathbf{F}$ (mass x acceleration = force)



$$\mathbf{F}_{em} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathbf{F}_g = M \mathbf{g}$$

Momentum equation for a fluid element: $\rho \frac{d\mathbf{v}}{dt} = \mathbf{f}$



mass density
(mass per unit volume)

acceleration

volume force (force per unit volume)

• Acceleration: $\frac{d\mathbf{v}}{dt}$... *Lagrangian derivative* $\Rightarrow \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}$... *Eulerian derivative*

• Volume force

gas pressure gradient force (collision-related): $\mathbf{f}_p = -\nabla p$

Coulomb force + Lorentz force (collision-unrelated; field-based): $\mathbf{f}_{em} = n q (\mathbf{E} + \mathbf{v} \times \mathbf{B})$

gravitational force (collision-unrelated; field-based): $\mathbf{f}_g = n M \mathbf{g}$

viscous force (collision-related): $\mathbf{f}_v = \rho \nu \left[\nabla^2 \mathbf{v} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{v}) \right]$, ν : viscosity

Lagrangian derivative vs. Eulerian derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$$

$$d A (x, y, z, t) = \frac{\partial A}{\partial x} d x + \frac{\partial A}{\partial y} d y + \frac{\partial A}{\partial z} d z + \frac{\partial A}{\partial t} d t$$

total differential

$$\begin{aligned} \therefore \frac{d A}{d t} &= \frac{\partial A}{\partial x} \frac{d x}{d t} + \frac{\partial A}{\partial y} \frac{d y}{d t} + \frac{\partial A}{\partial z} \frac{d z}{d t} + \frac{\partial A}{\partial t} \frac{d t}{d t} \\ &= \frac{\partial A}{\partial x} v_x + \frac{\partial A}{\partial y} v_y + \frac{\partial A}{\partial z} v_z + \frac{\partial A}{\partial t} \\ &= \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) A + \frac{\partial A}{\partial t} \\ &= \frac{\partial A}{\partial t} + (\mathbf{v} \cdot \nabla) A \end{aligned}$$

Momentum equation in MHD (simplified version... nonlinear term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ is neglected)

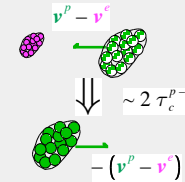
Proton's momentum equation (nonlinear term $(\mathbf{v}^p \cdot \nabla)\mathbf{v}^p$ is neglected):

$$n M \frac{\partial \mathbf{v}^p}{\partial t} = -\nabla P^p + n e (\mathbf{E} + \mathbf{v}^p \times \mathbf{B}) + \mathbf{f}_c^{p-e}$$

macroscale electric field
(global & continuous)

$$\mathbf{f}_c^{p-e} \approx -n M \frac{\mathbf{v}^p - \mathbf{v}^e}{\tau_c^{p-e}}$$

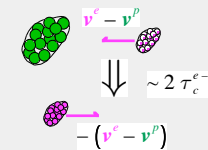
collision force based on microscale electric field (local & instantaneous)
=> change in momentum (relative velocity) per unit time



Electron's momentum equation (nonlinear term $(\mathbf{v}^e \cdot \nabla)\mathbf{v}^e$ is neglected):

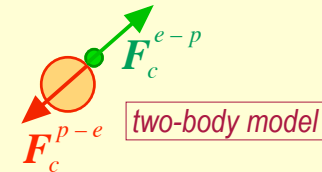
$$n m \frac{\partial \mathbf{v}^e}{\partial t} = -\nabla P^e - n e (\mathbf{E} + \mathbf{v}^e \times \mathbf{B}) + \mathbf{f}_c^{e-p}$$

$$\mathbf{f}_c^{e-p} \approx -n m \frac{\mathbf{v}^e - \mathbf{v}^p}{\tau_c^{e-p}}$$



Conservation of total momentum via collision

$$\mathbf{f}_c^{e-p} + \mathbf{f}_c^{p-e} = \mathbf{0} \left(\Rightarrow \tau_c^{e-p} = \frac{m}{M} \tau_c^{p-e} \right)$$



Add both equations...

$$n \frac{\partial}{\partial t} (M \mathbf{v}^p + m \mathbf{v}^e) = n e (\mathbf{v}^p - \mathbf{v}^e) \times \mathbf{B} - \nabla (P^p + P^e)$$

$$\rho^{MHD} \frac{\partial}{\partial t} \mathbf{v}^{MHD} = \mathbf{j}^{MHD} \times \mathbf{B} - \nabla P^{MHD}$$

$$P^{MHD} \equiv P^p + P^e: \text{total gas pressure}$$

$$\mathbf{j}^{MHD} \equiv n e (\mathbf{v}^p - \mathbf{v}^e): \text{current density}$$

Momentum eq. in MHD
(simplified version)