

MHD induction equation

... represents **electromagnetic part of MHD**

The diagram illustrates the derivation of the MHD induction equation. It starts with three fundamental laws on the left, each in a colored box: Faraday's law (purple), Ohm's law (brown), and Ampere's law (blue). Green arrows from Faraday's and Ohm's laws point to an intermediate equation, $\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (-\mathbf{v} \times \mathbf{B} + \eta \mathbf{j})$. A green arrow from Ampere's law points to a box labeled "eliminate $\mathbf{j}$ ", which then points down to the final MHD induction equation.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday's law}$$
$$\mathbf{E} = \eta \mathbf{j} - \mathbf{v} \times \mathbf{B} \quad \text{Ohm's law}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}^{\text{MHD}} \quad \text{Ampere's law}$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (-\mathbf{v} \times \mathbf{B} + \eta \mathbf{j})$$

eliminate $\mathbf{j}$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta_{\text{diff}} \nabla \times \mathbf{B})$$

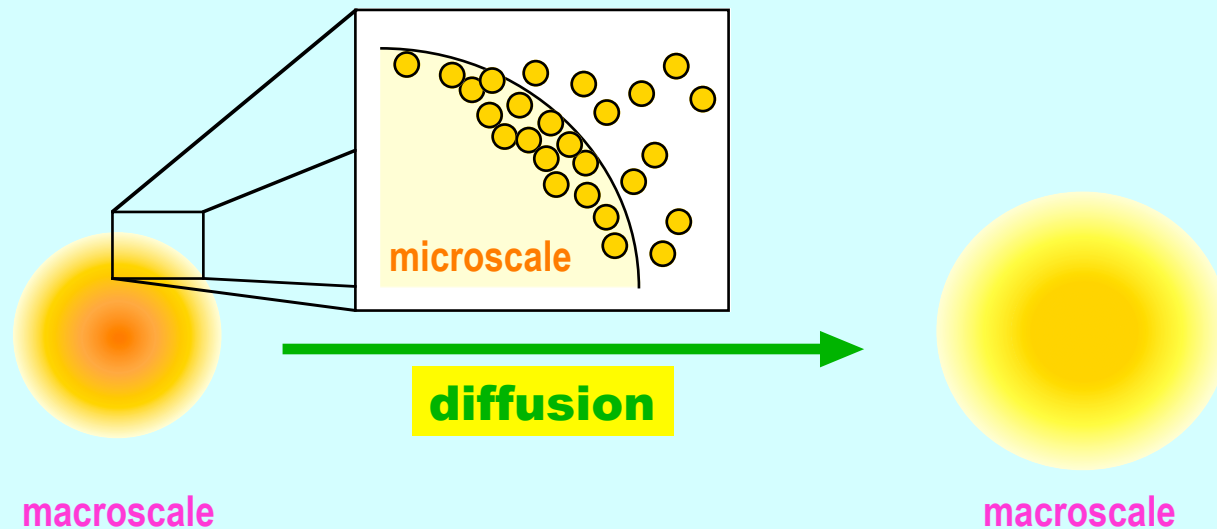
$$\eta_{\text{diff}} \equiv \frac{\eta}{\mu_0} \dots \text{magnetic diffusivity}$$

MHD induction equation

... describes **evolution of magnetic field** using **plasma quantities**

What is diffusion?

A **microscale process** ($\leq$ **kinetic approach**) caused by **collision of particles**, which makes a **macroscale distribution** ($\leq$ **fluid approach**) smooth.



Diffusion is characterized by **microscale**:

mean free path / inertial length l & **collision time τ_c**

dimension of diffusivity (m^2/s)... **velocity** (m/s) x **length** (m)

$$v l = \frac{l}{\tau_c} l = \frac{l^2}{\tau_c}$$

Magnetic diffusivity

Magnetic diffusivity

$$\eta_{diff} \equiv \frac{\eta}{\mu_0} = 5.2 \times 10^7 \frac{\ln N_D}{T^{3/2}} (\text{m}^2/\text{s}) \propto T^{-3/2}$$

$\ln N_D$ (Coulomb logarithm) ~ 10

$$v l = \frac{l}{\tau_c} \quad l = \frac{l^2}{\tau_c} \quad \text{where } l = \text{inertial length}$$

What diffuses is magnetic field.

Magnetic diffusivity is *lower* in a *hotter* region.

Electric energy density vs. Magnetic energy density

Electric energy density... $\mathcal{G}_E = \frac{\epsilon_0}{2} E^2$

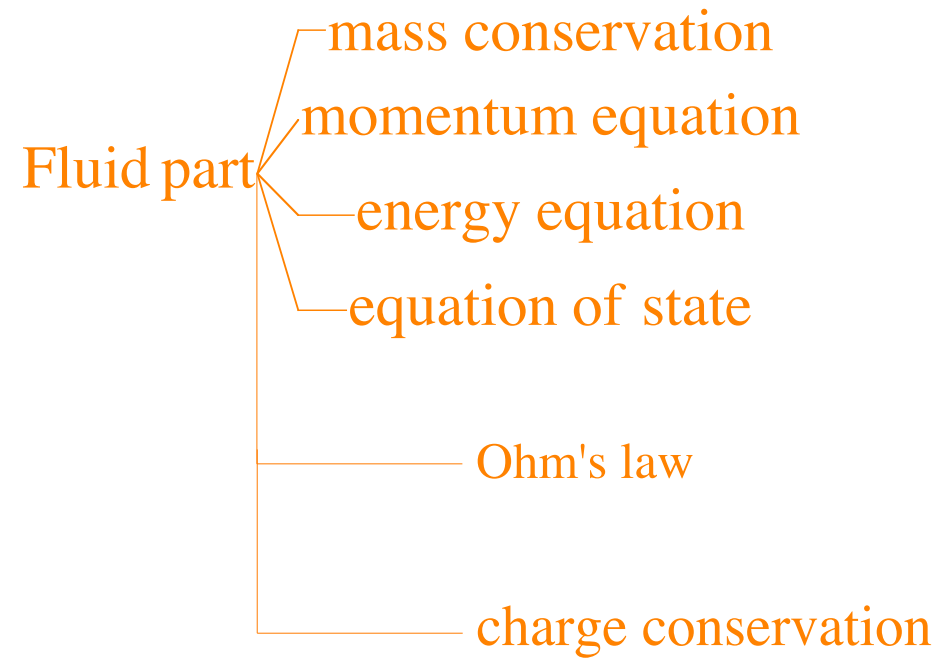
Magnetic energy density... $\mathcal{G}_M = \frac{B^2}{2 \mu_0}$

The ratio of them is given by

$$\nabla \times E = - \frac{\partial B}{\partial t} \rightarrow \frac{E_0}{l_0} \sim \frac{B_0}{t_0}$$

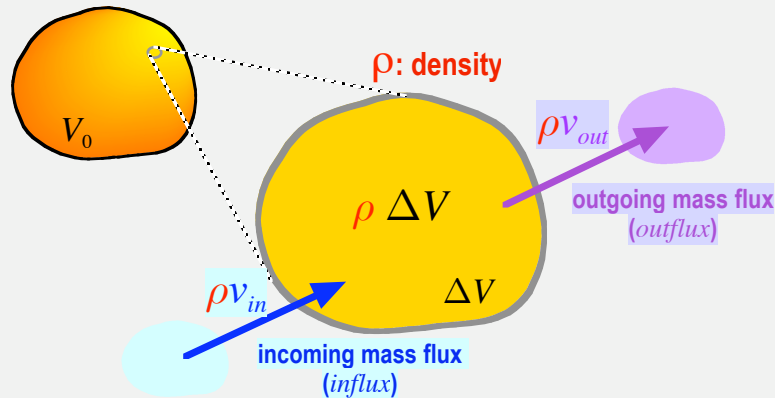
$$\mathcal{G}_E / \mathcal{G}_M = \frac{\epsilon_0 \mu_0 E^2}{B^2} \approx \frac{\left(\frac{l_0}{t_0} B_0 \right)^2}{c^2 B_0^2} = \frac{v_0^2}{c^2} \ll 1$$

→ Electric energy density is negligible when $v_0 \ll c$.



(Fields => Plasma)

Equation I... mass conservation



$$\frac{\Delta}{\Delta t} (\rho \Delta V) = \text{net influx} \equiv \text{influx} - \text{outflux}$$

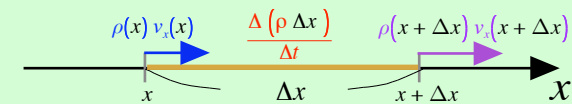
1D case

x and t are independent

$$\Delta x \frac{\Delta (\rho \Delta x)}{\Delta t} = \rho(x) v_x(x) - \rho(x + \Delta x) v_x(x + \Delta x)$$

$$\Rightarrow \frac{\Delta \rho}{\Delta t} = \frac{\rho(x) v_x(x) - \rho(x + \Delta x) v_x(x + \Delta x)}{\Delta x}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x} (\rho v_x)$$



3D case

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \mathbf{v}) \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\nabla \cdot (\rho \mathbf{v}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (\rho v_x, \rho v_y, \rho v_z)$$

$$= \frac{\partial (\rho v_x)}{\partial x} + \frac{\partial (\rho v_y)}{\partial y} + \frac{\partial (\rho v_z)}{\partial z}$$

$\nabla \cdot (\rho \mathbf{v})$... **net influx**; when this is positive, density decreases with time.



$$\nabla \cdot (\rho \mathbf{v}) > 0 \Rightarrow \frac{\partial \rho}{\partial t} < 0$$