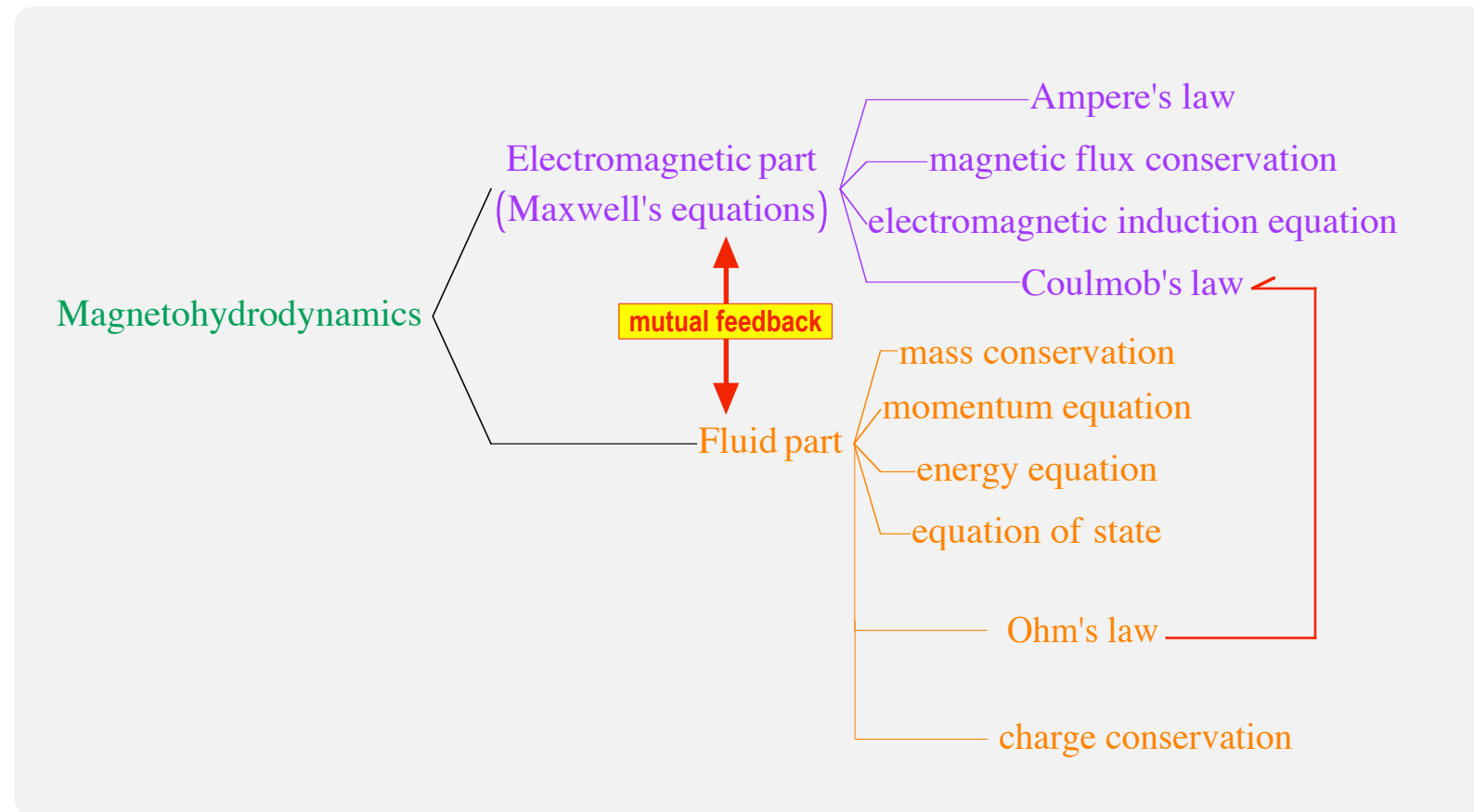


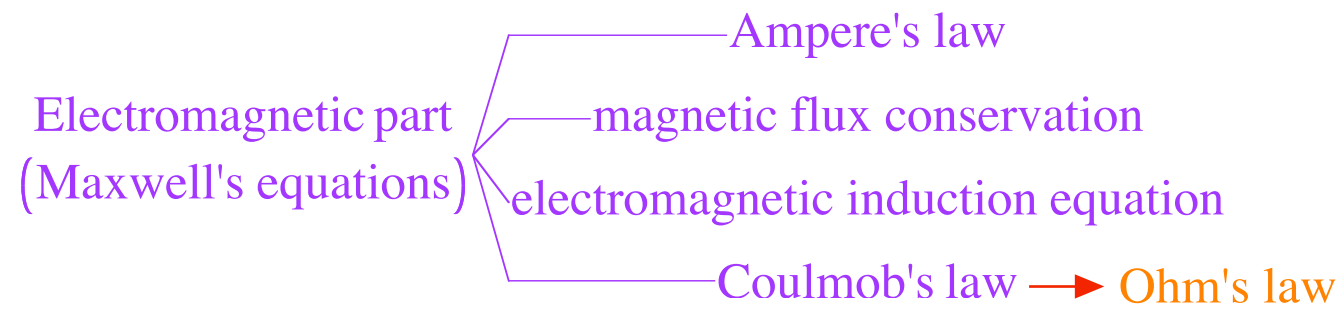
Magnetohydrodynamics

A fluid theory of plasmas

MHD describes...

how *plasma (fluid)* affects *electromagnetic fields*, while *electromagnetic fields* affect *plasma*.





(Plasma => Fields)

Maxwell's equations (determine electromagnetic fields from plasma quantities)

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \quad \dots \text{Generalized Ampere's law}$$

MKS unit

$$\nabla \cdot \mathbf{B} = 0 \quad \dots \text{Conservation of magnetic flux}$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad \dots \text{Electromagnetic induction equation (Faraday's law)}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \quad \dots \text{Coulomb's law}$$

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ F m}^{-1} \quad \dots \text{permittivity}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ H m}^{-1} \quad \dots \text{magnetic permeability}$$

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.998 \times 10^8 \text{ m s}^{-1} \quad \dots \text{speed of light}$$

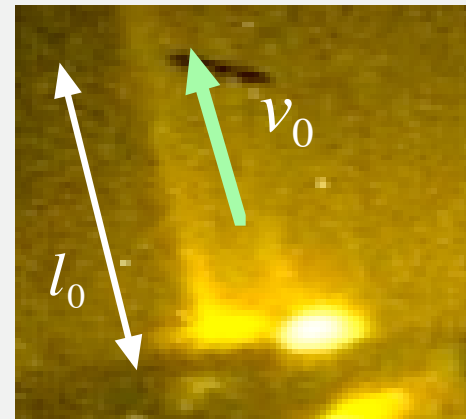
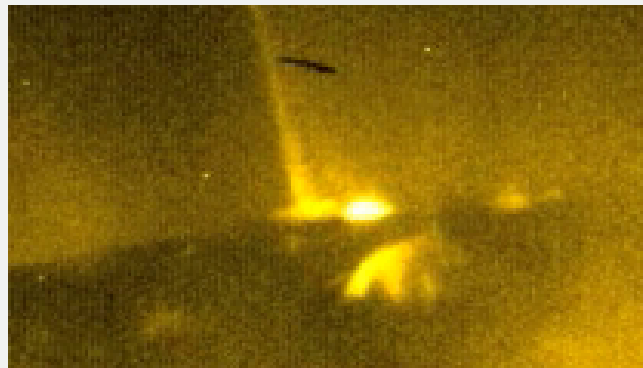
Unit:

$\mathbf{E}$... V/m, $\mathbf{B}$... T (= 10^4 G), $\mathbf{j}$... A/m²

Typical scale in MHD

MHD phenomena... *large-scale, slowly evolving phenomena*

e.g. *solar coronal jet*



Typical scale:

length... $l_0 \sim 10^4$ km

velocity... $v_0 \sim 100$ km/s

time... $t_0 = \frac{l_0}{v_0} \sim 100$ s

r_G (gyration radius of a proton) ~ 1 m
 τ_G (gyration period of a proton) $\sim 10^{-4}$ s
in the solar corona

Approximations in MHD

- $v_0 \ll c$ (non-relativistic approximation)

Comparison between *electromagnetic terms* in the *generalized Ampere's law*

$$\mathbf{j} = \frac{1}{\mu_0} \left(\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\frac{B_0}{l_0}$$

$$\frac{1}{c^2} \frac{E_0}{t_0}$$

Faraday's law

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \rightarrow \frac{E_0}{l_0} \sim \frac{B_0}{t_0}$$

$$\frac{1}{c^2} \frac{E_0}{t_0} \approx \frac{1}{c^2} \frac{1}{t_0} \frac{l_0}{t_0} B_0 = \frac{1}{c^2} \frac{l_0^2}{t_0^2} \frac{B_0}{l_0} = \frac{v_0^2}{c^2} \frac{B_0}{l_0} \ll \frac{B_0}{l_0}$$

$$\therefore \frac{B_0}{l_0} \gg \frac{1}{c^2} \frac{E_0}{t_0} \Rightarrow \text{we neglect } \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

$\Rightarrow$ Propagation of electromagnetic waves (radiation) is NOT described by MHD.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \Rightarrow \nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

Ampere's law in MHD

... determines **magnetic field** from **current density**

• $\rho_c \sim 0$ (local charge neutrality)

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \Rightarrow \frac{E_0}{l_0} \sim \frac{e(n_+ - n_-)}{\epsilon_0}$$

$$n_+ - n_- \sim \frac{\epsilon_0 E_0}{e l_0} \approx \frac{\epsilon_0 \left(\frac{l_0 B}{t_0} \right)}{e l_0} = \frac{\epsilon_0 v_0 B}{e l_0}$$

Condition of local charge neutrality:

$$n_+ - n_- \ll n_+ + n_- \equiv n_{total} \text{ (total number density)}$$

$$\therefore \frac{\epsilon_0 v_0 B}{e l_0} \ll n_{total} \longrightarrow 6.7 \times 10^7 \frac{v_0 B}{l_0} \ll n_{total}$$

$$e = 1.6 \times 10^{-19} \text{ C (electron charge)}$$

e.g. solar corona: $n_{total} \sim 10^{14} \text{ m}^{-3}$, $v_0 \sim 10^5 \text{ m/s}$, $l_0 \sim 10^7 \text{ m}$, $B \sim 10^{-2} \text{ T}$

$$6.7 \times 10^7 \frac{v_0 B}{l_0} \ll n_{total} \rightarrow 6.7 \times 10^3 \ll 10^{14} \dots \text{ satisfied}$$

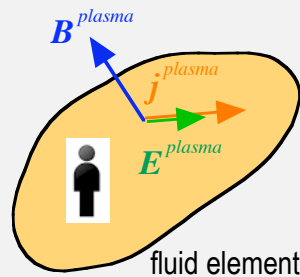
Coulomb's law in MHD $\nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0} \sim 0$

... does NOT determine electric field from charge density $\Rightarrow$ Ohm's law

Ohm's law

※ Rough explanation => more detailed explanation is given in fluid part

Plasma frame



$$\mathbf{E}^{plasma} = \eta \mathbf{j}^{plasma} = \frac{1}{\sigma} \mathbf{j}^{plasma}$$

Ohm's law (plasma frame)

Electric field is parallel to electric current density.

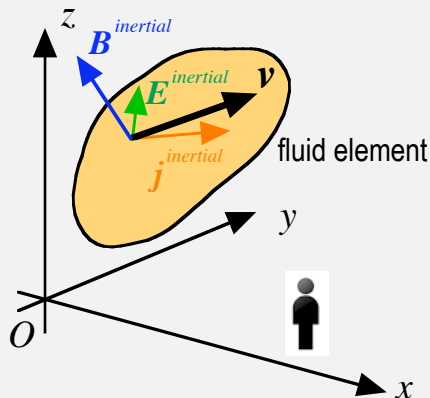
η ... resistivity (ohm m = kg m³ s⁻³ A⁻²)

σ ... electric conductivity (ohm⁻¹ m⁻¹ = mho m⁻¹)

=> microscale quantities determined by kinetic approach

$$\sigma = 1.53 \times 10^{-2} \frac{T^{3/2}}{\ln N_D} \text{ (mho/m)} \propto T^{3/2}$$

Inertial frame



When a fluid element moves at $\mathbf{v}$ in inertial frame, then

$$\mathbf{j}^{inertial} = \mathbf{j}^{plasma}$$

$$\mathbf{E}^{inertial} = \mathbf{E}^{plasma} - \mathbf{v} \times \mathbf{B}^{plasma} \quad \mathbf{B}^{inertial} = \mathbf{B}^{plasma}$$

(non-relativistic, charge neutral approximations)

So,

$$\mathbf{E}^{inertial} = \eta \mathbf{j}^{inertial} - \mathbf{v} \times \mathbf{B}^{inertial}$$

Ohm's law (inertial frame)

... determines electric field from flow velocity & current density

$$\mathbf{E}_{convective} \equiv -\mathbf{v} \times \mathbf{B}^{inertial} \quad \dots \text{ transport magnetic field at } \mathbf{v}_{\perp}$$