

4. Integral transform method

$$u(x, t) \rightarrow U(k, t) = \int u(x, t) \boxed{e^{-ikx}} dx \dots \text{Fourier transform}$$

$$\text{e.g., } \frac{\partial u(x, t)}{\partial t} = \alpha \frac{\partial^2}{\partial x^2} u(x, t) \rightarrow \int \frac{\partial u(x, t)}{\partial t} e^{-ikx} dx = \int \alpha \frac{\partial^2}{\partial x^2} u(x, t) e^{-ikx} dx$$

$$\rightarrow \frac{dU(k, t)}{dt} = -\alpha k^2 U(k, t) \dots \text{ODE}$$

$$u(x, t) \rightarrow U(x, p) = \int_0^\infty u(x, t) \boxed{e^{-pt}} dt \dots \text{Laplace transform}$$

$$\text{e.g., } \frac{\partial u(x, t)}{\partial t} = \alpha \frac{\partial^2}{\partial x^2} u(x, t) \rightarrow \int_0^\infty \frac{\partial u(x, t)}{\partial t} e^{-pt} dt = \int_0^\infty \alpha \frac{\partial^2}{\partial x^2} u(x, t) e^{-pt} dt$$

$$\rightarrow p U(x, p) - \varphi(x) = \alpha \frac{d^2}{dx^2} U(x, p) \dots \text{ODE}$$

$$\varphi(x) = u(x, 0): \text{initial condition}$$

$$e^{-ikx}, e^{-pt} \dots \text{kernel of integral transform}$$

Using integral transform, PDE is transformed to ODE.

5. Self-similar analysis

e.g., $\frac{\partial}{\partial t} T - \alpha \nabla^2 T = 0$

spherically symmetric
(isotropic) case
 $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}$

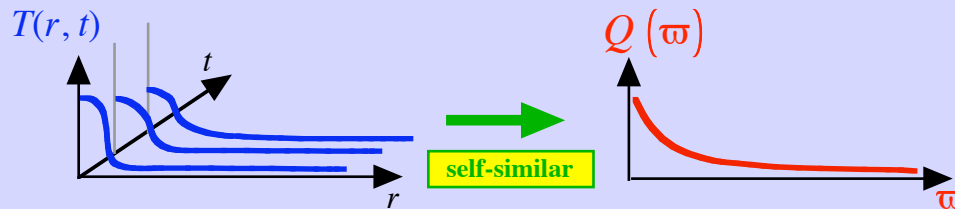
$$\frac{\partial}{\partial t} T(r, t) - \alpha \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right] T(r, t) = 0 \dots \text{PDE}$$

Transform*: $T(r, t) \rightarrow Q(\varpi)$

*Unlike Fourier and Laplace transforms, a way of transform is found out heuristically.

e.g., $\varpi = \frac{r^2}{\alpha t}, Q = \frac{T(r, t)}{q} C(\alpha t)^{\frac{3}{2}}$ q : total heat, C : heat capacity... const.

ϖ, Q ... dimensionless quantities $\Rightarrow Q$ could be given by a function of ϖ : $Q = Q(\varpi)$



Function with Two Variables is **self-similar** to Function with a Single Variable

Partial derivative

$\Rightarrow$ **Total derivative**

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial^2}{\partial r^2} \rightarrow \frac{d}{d\varpi}, \frac{d^2}{d\varpi^2}$$

$$\frac{\partial T}{\partial t} = \frac{q}{C(\alpha t)^{\frac{3}{2}}} \left\{ -\frac{3}{2t} Q - \frac{\varpi}{t} \frac{dQ}{d\varpi} \right\}$$

$$\frac{\partial T}{\partial r} = \frac{q}{C(\alpha t)^{\frac{3}{2}}} \frac{2r}{\alpha t} \frac{dQ}{d\varpi}$$

$$\frac{\partial^2 T}{\partial r^2} = \frac{q}{C(\alpha t)^{\frac{3}{2}}} \left\{ \frac{2}{\alpha t} \frac{dQ}{d\varpi} - \frac{4r^2}{\alpha^2 t^2} \frac{d^2 Q}{d\varpi^2} \right\}$$

$$4\varpi \frac{d^2 Q}{d\varpi^2} + (\varpi + 6) \frac{dQ}{d\varpi} + \frac{3}{2} Q(\varpi) = 0 \dots \text{ODE}$$

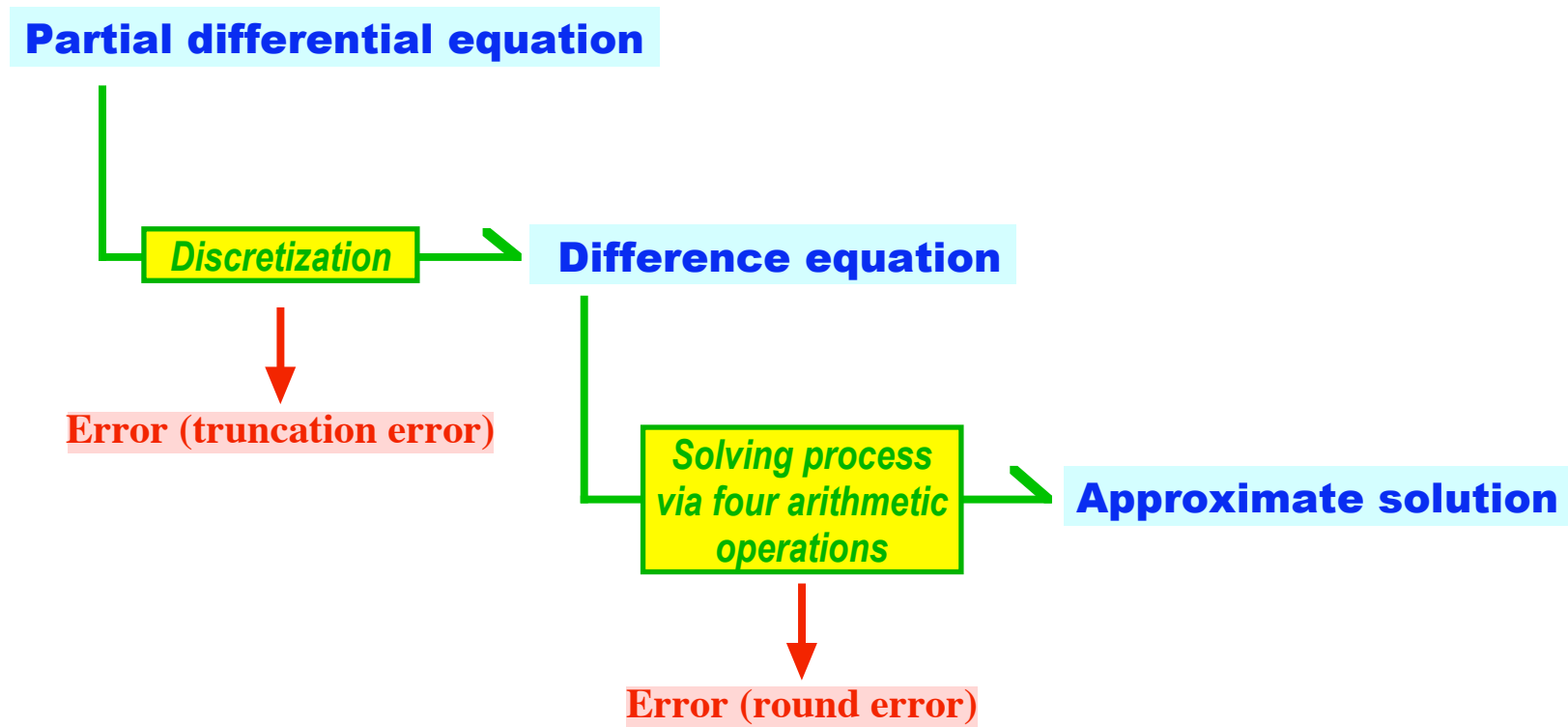
$$Q(\varpi) \propto e^{-\frac{\varpi}{4}}$$

Using **self-similar analysis**, PDE is transformed to **ODE**.

Numerical scheme for diffusion equation

$$\frac{\partial}{\partial t} T - \alpha \frac{\partial^2}{\partial x^2} T = 0$$

Flowchart of numerical simulation:



Discretization

Model equation

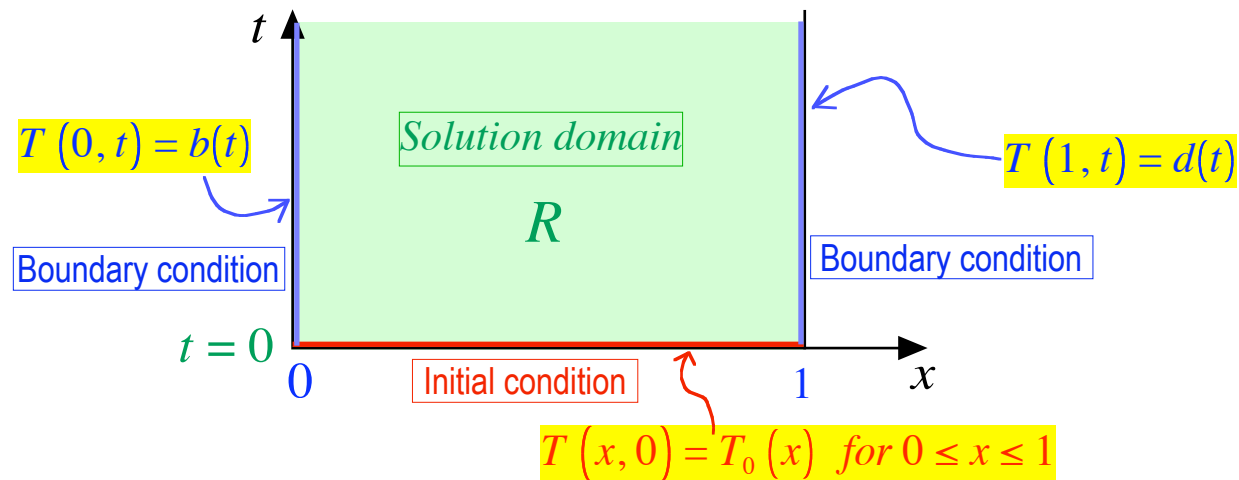
1-dimensional thermal conduction equation (*parabolic PDE*):

$$\frac{\partial}{\partial t} T - \alpha \frac{\partial^2}{\partial x^2} T = 0$$

α : thermal diffusivity

Boundary condition: $T(0, t) = b(t)$, $T(1, t) = d(t)$

Initial condition: $T(x, 0) = T_0(x)$ for $0 \leq x \leq 1$

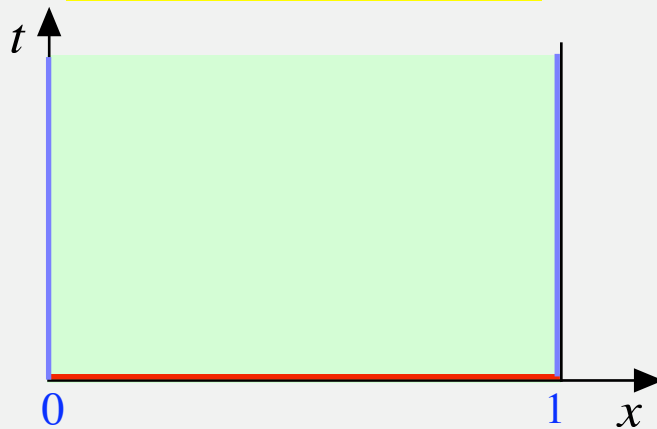


Differential equation and Difference equation

Differential equation

$$\frac{\partial}{\partial t} T - \alpha \frac{\partial^2}{\partial x^2} T = 0$$

continuous solution domain



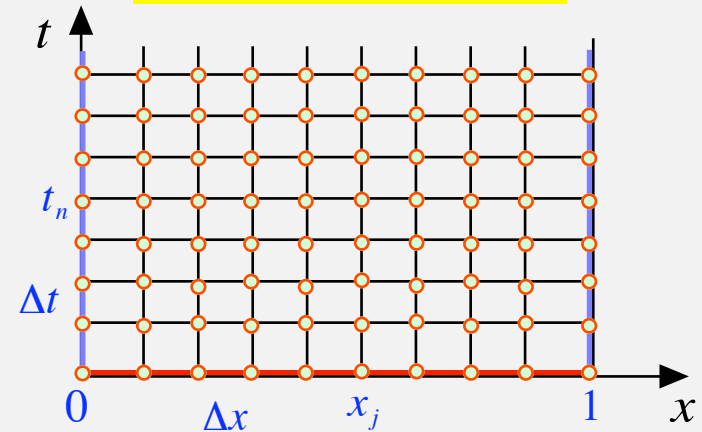
$$T = T(x, t)$$

T is obtained at **any** (x, t) .

Difference equation

$$\frac{T_j^{n+1} - T_j^n}{\Delta t} - \alpha \frac{T_{j-1}^n - 2T_j^n + T_{j+1}^n}{\Delta x^2} = 0$$

discrete solution domain



$$T = T(x_j, t_n) = T_j^n$$

T is obtained only at **grid points** (x_j, t_n) .

$$j = 1, 2, \dots, j_{\max}; n = 0, 1, 2, \dots, n_{\max}$$

How to convert differential to difference (finite-difference method)

Taylor expansion

$$T(x_{j+1}, t_n) = T(x_j, t_n) + \frac{\partial T}{\partial x}(x_j, t_n) \Delta x + \frac{\partial^2 T}{\partial x^2}(x_j, t_n) \frac{\Delta x^2}{2} + O(\Delta x^3) \dots \text{for space}$$

$$T(x_j, t_{n+1}) = T(x_j, t_n) + \frac{\partial T}{\partial t}(x_j, t_n) \Delta t + \frac{\partial^2 T}{\partial t^2}(x_j, t_n) \frac{\Delta t^2}{2} + O(\Delta t^3) \dots \text{for time}$$

$O(\Delta x^3), O(\Delta t^3) \dots$ 3rd and even higher-order terms of Δx and Δt

$$\frac{\partial T}{\partial x}(x_j, t_n) = \frac{T(x_{j+1}, t_n) - T(x_j, t_n)}{\Delta x} - \frac{\partial^2 T}{\partial x^2}(x_j, t_n) \frac{\Delta x}{2} + O(\Delta x^2)$$

leading term
(1st-order)

2nd and even higher-
order terms

Truncation error

$$\frac{\partial T}{\partial x}(x_j, t_n) \approx \frac{T(x_{j+1}, t_n) - T(x_j, t_n)}{\Delta x}$$

... Forward-difference method (1st-order approximation)

High-order approximation

3-point formula

$$T_{j+1}^n = T_j^n + \frac{\partial T}{\partial x}(x_j, t_n) \Delta x + \frac{\partial^2 T}{\partial x^2}(x_j, t_n) \frac{\Delta x^2}{2} + \frac{\partial^3 T}{\partial x^3}(x_j, t_n) \frac{\Delta x^3}{6} + O(\Delta x^4)$$

$$T_j^n = T_j^n$$

$$T_{j-1}^n = T_j^n - \frac{\partial T}{\partial x}(x_j, t_n) \Delta x + \frac{\partial^2 T}{\partial x^2}(x_j, t_n) \frac{\Delta x^2}{2} - \frac{\partial^3 T}{\partial x^3}(x_j, t_n) \frac{\Delta x^3}{6} + O(\Delta x^4)$$



$$\begin{aligned} a T_{j-1}^n + b T_j^n + c T_{j+1}^n &= (a + b + c) T_j^n + (-a + c) \frac{\partial T}{\partial x}(x_j, t_n) \Delta x + (a + c) \frac{\partial^2 T}{\partial x^2}(x_j, t_n) \frac{\Delta x^2}{2} \\ &\quad + (-a + c) \frac{\partial^3 T}{\partial x^3}(x_j, t_n) \frac{\Delta x^3}{6} + O(\Delta x^4) \end{aligned}$$

when $a = -c = \frac{1}{2}, b = 0 \Rightarrow$

$$\frac{\partial T}{\partial x}(x_j, t_n) = \frac{T_{j+1}^n - T_{j-1}^n}{2 \Delta x} - \frac{\partial^3 T}{\partial x^3}(x_j, t_n) \frac{\Delta x^2}{6} + \dots$$

leading term (2nd-order)

... Centered-difference method (2nd-order approximation)

Comparison between forward, backward, and centered-difference methods

