

Characteristics of parabolic PDE

Example of parabolic PDE:

Diffusion equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$y \rightarrow t, A = -1, B = 0, C = 0$$

$$B^2 - 4AC = 0 - 4 \times (-1) \times 0 = 0$$

characteristic equation

$$\left[A \left(\frac{dy}{dx} \right)^2 - B \left(\frac{dy}{dx} \right) + C \right] = 0 \quad \longrightarrow \quad \frac{dt}{dx} = 0$$

$$y \rightarrow t, A = -1, B = 0, C = 0$$

There is **one family of characteristics**: $\frac{dt}{dx} = 0$

On the characteristics $K: t = \text{const}$,

$$\left[A \left(\frac{du_x}{dx} \right) \right] \frac{dy}{dx} + C \frac{du_y}{dx} = 0$$

$$y \rightarrow t, A = -1, B = 0, C = 0$$

$$\left[- \left(\frac{du_x}{dx} \right) \right] \frac{dt}{dx} = 0$$

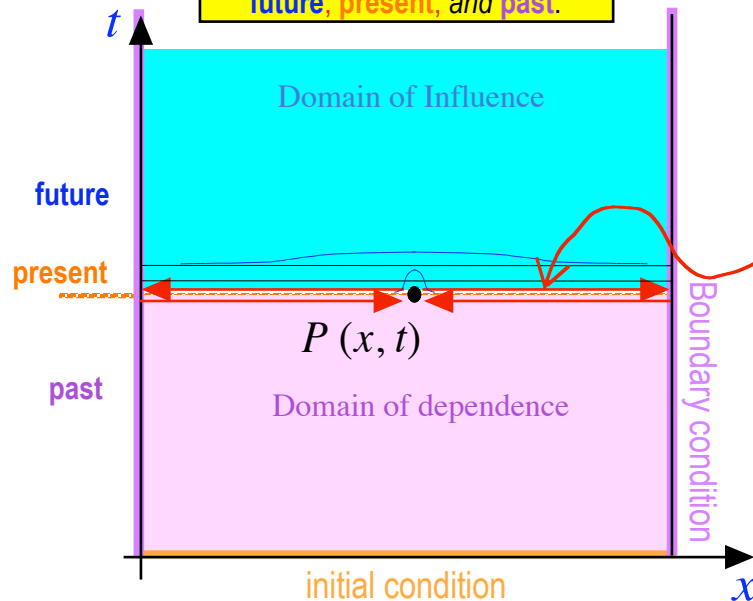
$$\frac{dt}{dx} = 0 \quad \text{characteristic equation}$$

On the characteristics $K: t = \text{const}$,

$$0 = 0 \quad \text{with} \quad \frac{du_x(x, t = \text{const})}{dx} \neq \infty$$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

Solution domain is divided into future, present, and past.



The information provided by initial condition is spread over the whole spatial range instantaneously.

P depends on the whole past domain.
 P influences the whole future domain.

Information is transported between different spatial locations without any delay (no propagation).

=> P depends on the whole past domain, while the whole future domain depends on P (domain of dependence and domain of influence are NOT limited).

Characteristics of elliptic PDE

Example of elliptic PDE:

Laplace equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$A = 1, B = 0, C = 1$$

$$B^2 - 4AC = 0 - 4 \times 1 \times 1 = -4 < 0$$

characteristic equation

$$\left[A \left(\frac{dy}{dx} \right)^2 - B \left(\frac{dy}{dx} \right) + C \right] = 0$$

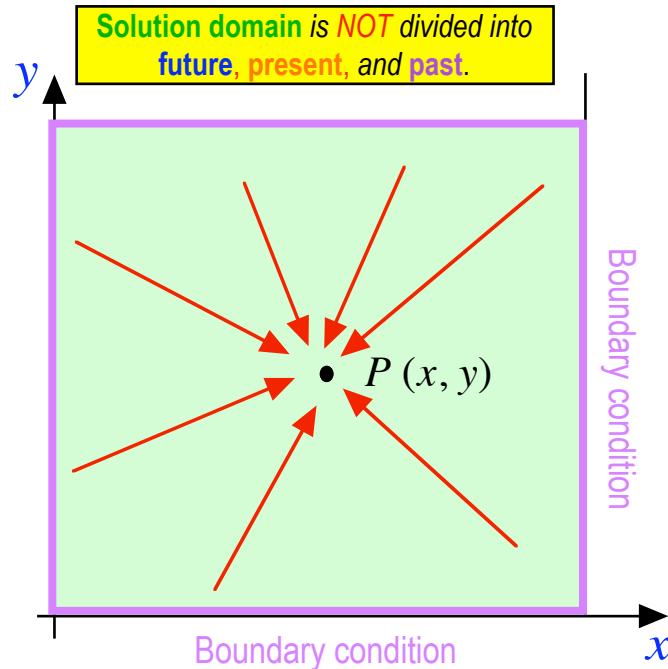


$$A = 1, B = 0, C = 1$$

$$\left(\frac{dy}{dx} \right)^2 = -1$$

There is **no family of characteristics**.

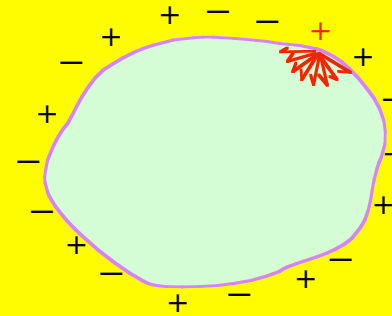
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



Information provided by boundary condition determines and fixes one steady solution in the whole solution domain.

*No characteristics
=> no specific direction of transporting information*

Information spreads from each location on the boundary in all directions.

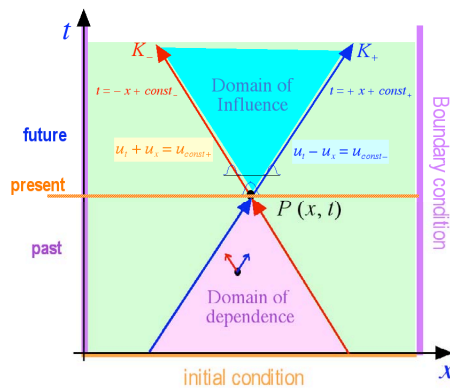


"time" is not introduced to elliptic PDEs.

=> steady or static

Hyperbolic PDE

Solution domain is divided into future, present, and past.

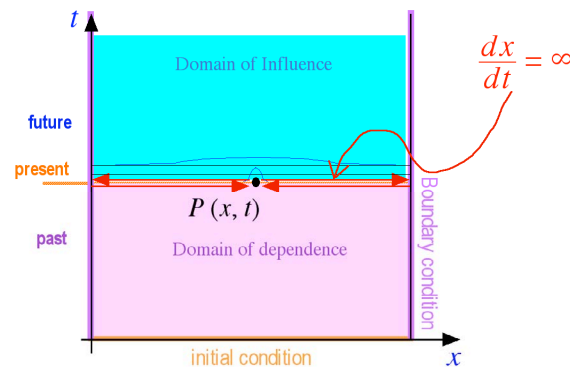


P depends on **part of** past domain.
 P influences **part of** future domain.

Two families of characteristics

Parabolic PDE

Solution domain is divided into future, present, and past.

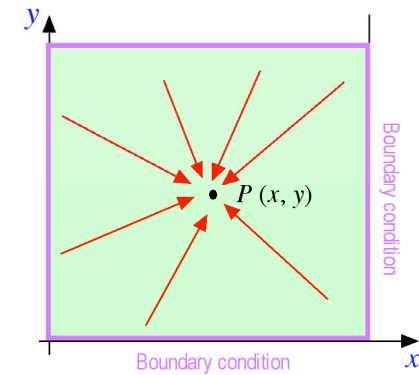


P depends on the **whole** past domain.
 P influences the **whole** future domain.

One family of characteristics

Elliptic PDE

Solution domain is **NOT** divided into future, present, and past.

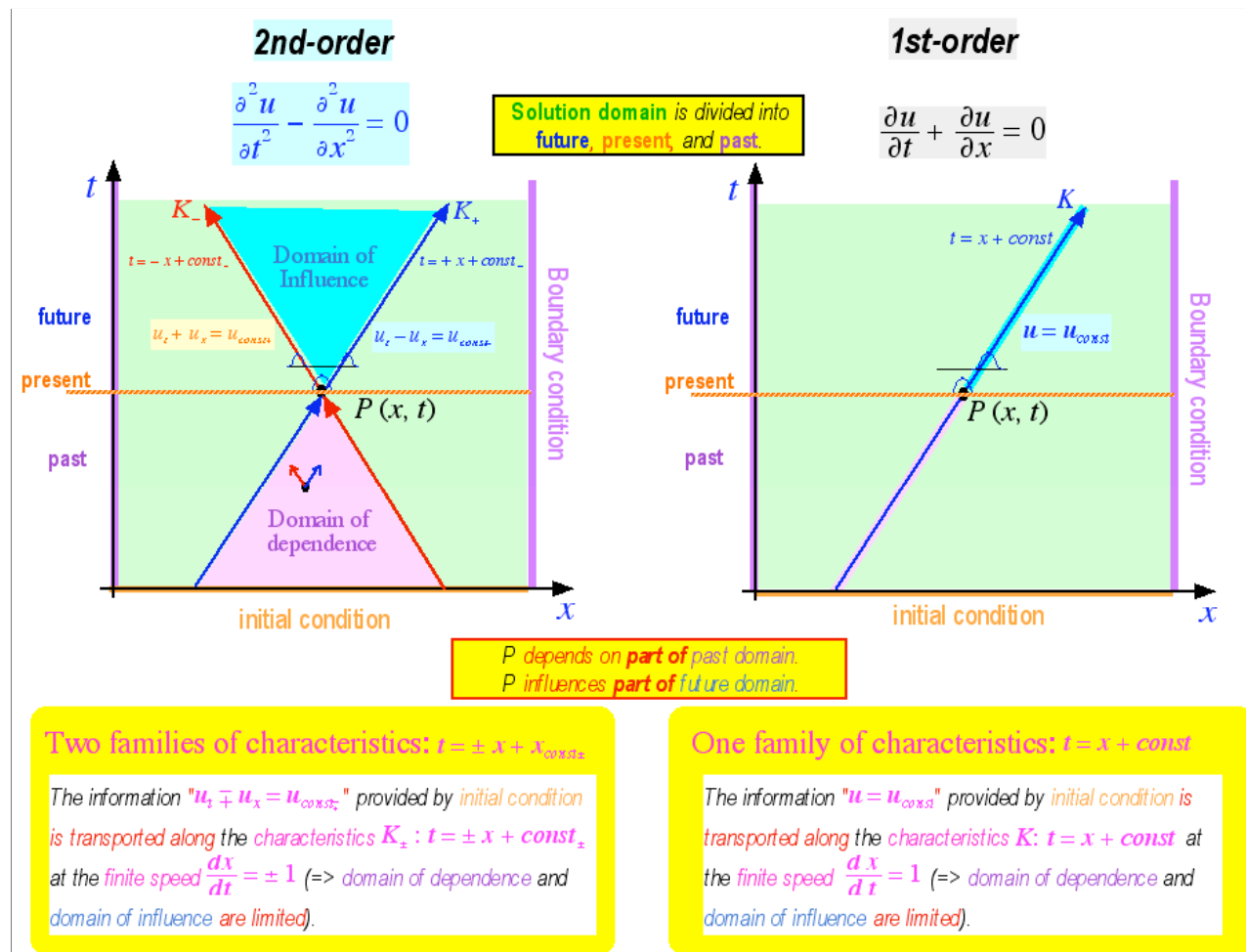


Information provided by **boundary condition** determines and fixes one steady solution in the **whole** solution domain.

No family of characteristics

Traditional analytical methods for solving PDEs

1. Characteristics method



applicable to hyperbolic PDEs

2. Separation of Variables

e.g.,

$$\frac{\partial u(x, t)}{\partial t} - \alpha \frac{\partial^2}{\partial x^2} u(x, t) = 0$$

$$u(x, t) = X(x) T(t)$$

Assume that solution function is given by a product of x -function and t -function (separation of variables).

$$\begin{cases} \frac{d^2 X(x)}{dx^2} - \frac{\lambda}{\alpha} X(x) = 0 \\ \frac{dT(t)}{dt} - \lambda T(t) = 0 \end{cases} \quad \lambda = \text{const.}$$

PDE $\Rightarrow$ a set of ODEs

applicable to the three types of PDEs
(as long as the assumption is valid)

3. Green's function method

