

Key point... PDE is converted to a set of ODEs

Partial differential equation in (x, t) -plane:

$$A \frac{\partial u}{\partial t} + B \frac{\partial u}{\partial x} = C$$



$$u = u(x, t)$$

solution surface



A set of ordinary differential equations on s -curve:

$$\begin{cases} \frac{dt}{ds} = A \\ \frac{dx}{ds} = B \\ \frac{du}{ds} = C \end{cases}$$

characteristics

$$A = A(t(s), x(s)) = A(s)$$

$$B = B(t(s), x(s)) = B(s)$$

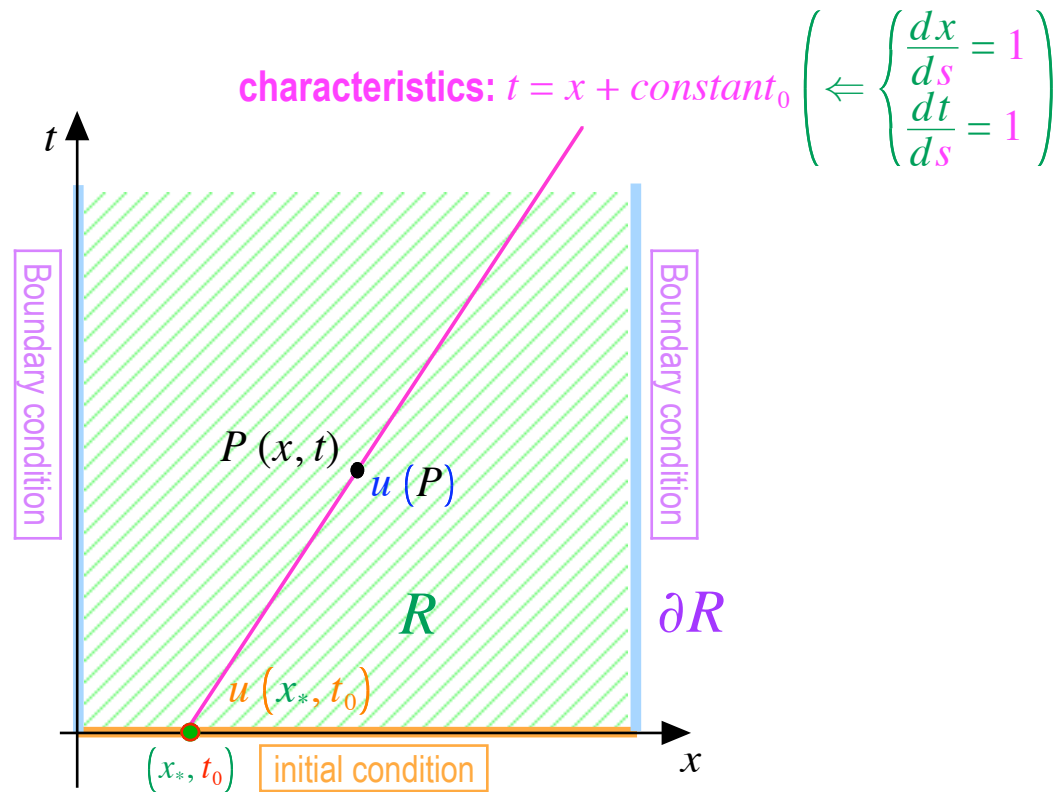
$$C = C(t(s), x(s)) = C(s)$$



$$(t(s), x(s), u(s))$$

a group of solution curves

Example: $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \longrightarrow \frac{dt}{ds} = A = 1, \frac{dx}{ds} = B = 1, \frac{du}{ds} = C = 0$



On $t = x + \text{constant}_0$, $u = \text{constant}_1 \left(\Leftrightarrow \frac{du}{ds} = 0 \right)$

$\Rightarrow u(P) = u(x_*, t_0) \dots$ this is given by **initial condition**

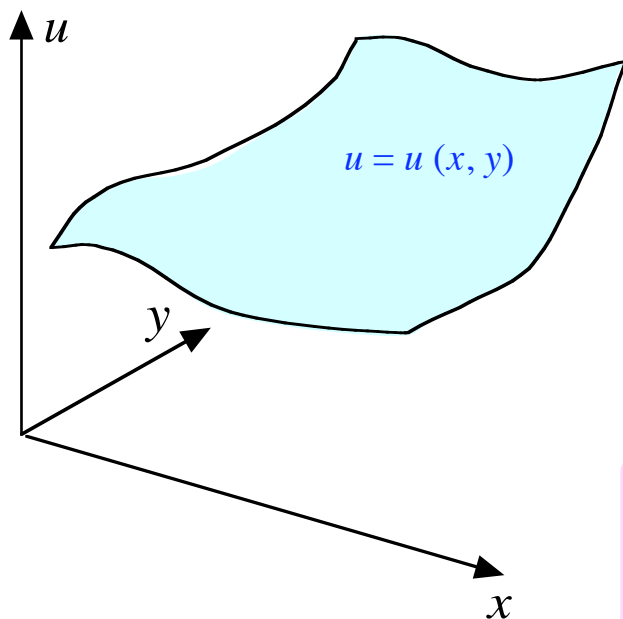
Characteristics of 2nd-order linear PDE

2nd-order linear PDE with two independent variables

(only 2nd-order derivative terms are taken into account)

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0$$

$$\longrightarrow A u_{xx} + B u_{xy} + C u_{yy} = 0$$



$$du_x = u_{xx} dx + u_{xy} dy$$

$$du_y = u_{yx} dx + u_{yy} dy$$

u_{xx} and u_{yy} are
eliminated

$u_{xy} = u_{yx}$ is
assumed

$$u_{xy} \left[A \left(\frac{dy}{dx} \right)^2 - B \left(\frac{dy}{dx} \right) + C \right] - \left\{ \left[A \left(\frac{du_x}{dx} \right) \right] \frac{dy}{dx} + C \frac{du_y}{dx} \right\} = 0$$

$$u_{xy} \left[A \left(\frac{dy}{dx} \right)^2 - B \left(\frac{dy}{dx} \right) + C \right] - \left\{ \left[A \left(\frac{du_x}{dx} \right) \right] \frac{dy}{dx} + C \frac{du_y}{dx} \right\} = 0$$

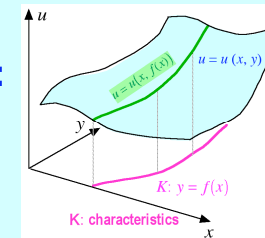
Since $u_{xy}(x, y)$ is generally a *nonzero function* of (x, y) ,

$$A \left(\frac{dy}{dx} \right)^2 - B \left(\frac{dy}{dx} \right) + C = 0 \Rightarrow \text{characteristic equation} \quad \text{quadratic equation}$$

This equation determines **characteristics** $K: y = f(x)$ in solution domain.

On K , PDE is reduced to the following ODE (x is the only independent variable):

$$\left[A \left(\frac{du_x}{dx} \right) \right] \frac{dy}{dx} + C \frac{du_y}{dx} = 0 \quad \text{with } y = f(x)$$



Discriminant

Hyperbolic PDE: $B^2 - 4AC > 0 \Rightarrow$ **two families** of characteristics

Parabolic PDE: $B^2 - 4AC = 0 \Rightarrow$ **one family** of characteristics

Elliptic PDE: $B^2 - 4AC < 0 \Rightarrow$ **no family** of characteristics

Characteristics of hyperbolic PDE

Example of hyperbolic PDE:

Wave equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$y \rightarrow t, A = -1, B = 0, C = 1$$

$$B^2 - 4AC = 0 - 4 \times (-1) \times 1 = 4 > 0$$

characteristic equation

$$\left[A \left(\frac{dy}{dx} \right)^2 - B \left(\frac{dy}{dx} \right) + C \right] = 0 \quad \longrightarrow \quad \left(\frac{dt}{dx} \right)^2 = 1$$

$$y \rightarrow t, A = -1, B = 0, C = 1$$

There are **two families** of characteristics: $\frac{dt}{dx} = \pm 1$

On the characteristics $K_{\pm} : t = \pm x + \text{const}_{\pm}$,

$$\left[A \left(\frac{du_x}{dx} \right) \right] \frac{dy}{dx} + C \frac{du_y}{dx} = 0$$



$$y \rightarrow t, A = -1, B = 0, C = 1$$

$$\left[- \left(\frac{du_x}{dx} \right) \right] \frac{dt}{dx} + \frac{du_t}{dx} = 0$$



$$\frac{dt}{dx} = \pm 1 \quad \text{characteristic equation}$$

$$\frac{d(u_t \mp u_x)}{dx} = 0$$



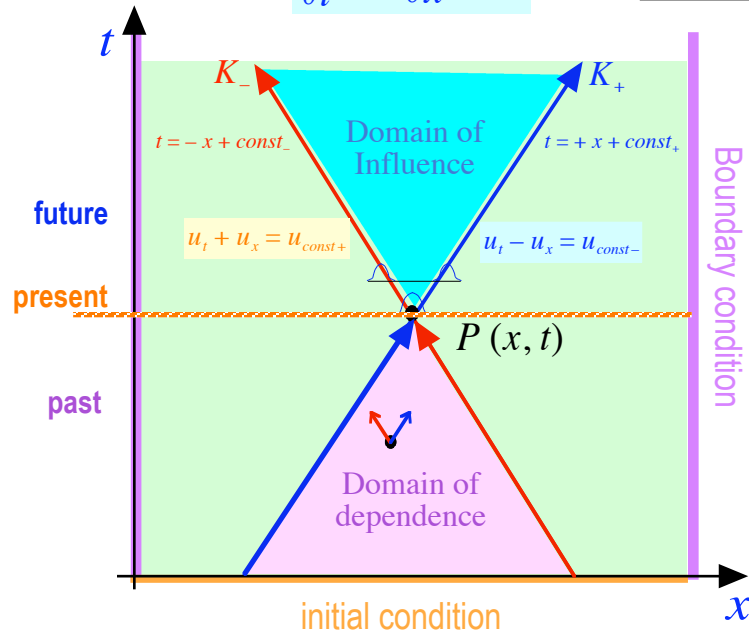
On the characteristics $K_{\pm} : t = \pm x + \text{const}_{\pm}$,

$$u_t \mp u_x = u_{\text{const}_{\pm}}$$

2nd-order

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

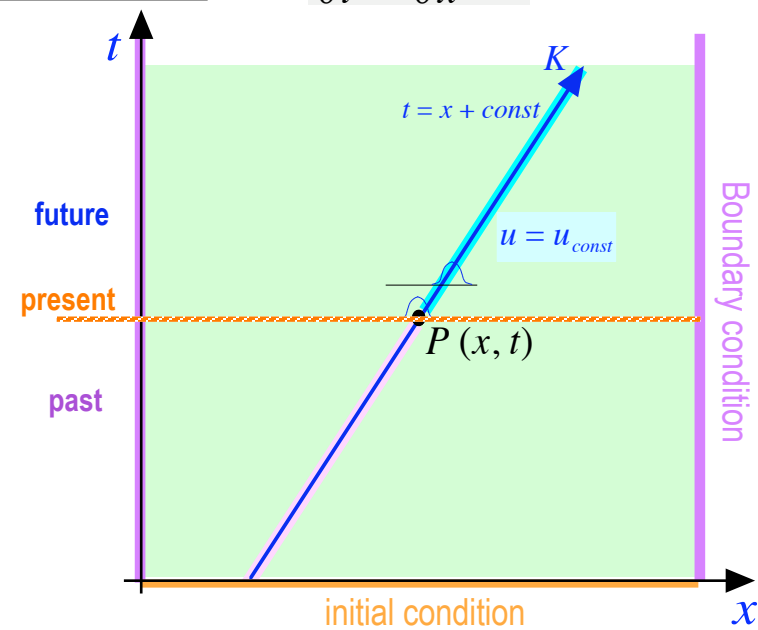
Solution domain is divided into future, present, and past.



*P depends on part of past domain.
P influences part of future domain.*

1st-order

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$



One family of characteristics: $t = x + \text{const}$

The information " $u = u_{\text{const}}$ " provided by initial condition is transported along the characteristics $K: t = x + \text{const}$ at the finite speed $\frac{dx}{dt} = 1$ ($\Rightarrow$ domain of dependence and domain of influence are limited).

Two families of characteristics: $t = \pm x + \text{const}$

The information " $u_t \mp u_x = u_{\text{const}\pm}$ " provided by initial condition is transported along the characteristics $K_{\pm}: t = \pm x + \text{const}_{\pm}$ at the finite speed $\frac{dx}{dt} = \pm 1$ ($\Rightarrow$ domain of dependence and domain of influence are limited).