

Characteristics

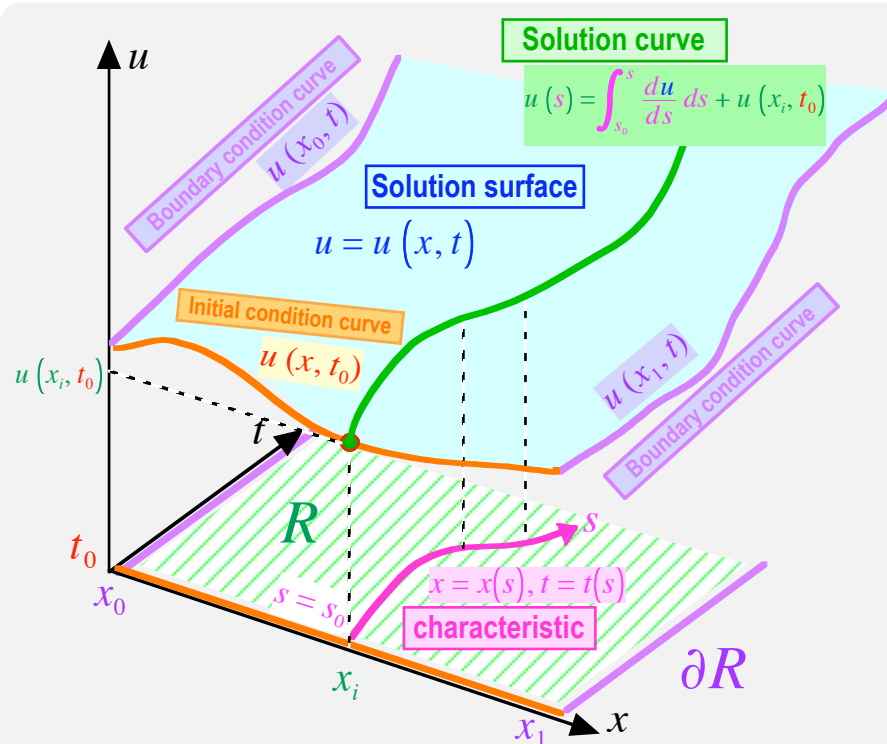
Characteristics of 1st-order linear PDE: $A \frac{\partial u}{\partial t} + B \frac{\partial u}{\partial x} = C$

$u = u(x, t)$... solution function

Characteristics $\begin{cases} x = x(s) \\ t = t(s) \end{cases}$... special curves in (x, t) -plane or R

Solution function can be derived via direct integration along characteristics $x = x(s, x_i, t_0), t = t(s, x_i, t_0)$:

$$u = \int du = \int_{x\text{-axis}} \frac{\partial u}{\partial x} dx + \int_{t\text{-axis}} \frac{\partial u}{\partial t} dt = \int_{\text{characteristic}} \left(\frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial t} \frac{dt}{ds} \right) ds = \int_{\text{characteristic}} \frac{du}{ds} ds$$



$u = u(x, t)$ forms solution surface in (x, t, u) -space.

Characteristic gives a solution curve $u(s)$ in solution surface.

$$u(s) = \int_{s_0}^s \frac{du}{ds} ds + u(x_i, t_0)$$

Solution surface

= **Group of Solution curves**

$$u(s) = \int_{s_0}^s \frac{du}{ds} ds + u(x_i, t_0) \quad x_0 \leq x_i \leq x_1$$

How to derive characteristics

$$A \frac{\partial u}{\partial t} + B \frac{\partial u}{\partial x} = C$$



$$(A, B, C) \perp \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1 \right) \text{ because the inner product of these vectors is zero.}$$

$$(A, B, C) \cdot \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1 \right) = 0$$

$\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1 \right)$ is a **normal vector** on solution surface $u = u(x, t)$,
because ...

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx \Rightarrow \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx + (-1) du = 0$$

$$\Rightarrow \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1 \right) \cdot (dt, dx, du) = 0 \Rightarrow \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1 \right) \perp (dt, dx, du)$$

where (dt, dx, du) is any tangential vector on solution surface.

$$(A, B, C) \perp \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, -1 \right) \Rightarrow (A, B, C) \text{ is a special tangential vector on solution surface.}$$



$$\frac{dt}{A} = \frac{dx}{B} = \frac{du}{C} = ds \text{ determines solution curves } (t(s), x(s), u(s)).$$

$$\left(\frac{dt}{A} = \frac{dx}{B} = ds \text{ determines characteristics.} \right)$$