

Three types of 2nd-order linear PDEs

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Hyperbolic PDE... $\alpha^2 \frac{\partial^2 u}{\partial x^2} - \beta^2 \frac{\partial^2 u}{\partial y^2} = f \Leftrightarrow \alpha^2 x^2 - \beta^2 y^2 = \gamma^2$

Hyperbolic curve

$\Rightarrow$ Wave equation: $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ (**time-dependent**)

Parabolic PDE... $\alpha^2 \frac{\partial u}{\partial x} - \beta^2 \frac{\partial^2 u}{\partial y^2} = f \Leftrightarrow \alpha^2 x - \beta^2 y^2 = \gamma^2$

Parabolic curve

$\Rightarrow$ Diffusion equation: $\frac{\partial T}{\partial t} - \alpha \frac{\partial^2}{\partial x^2} T = 0$ (**time-dependent**)

Elliptic PDE... $\alpha^2 \frac{\partial^2 u}{\partial x^2} + \beta^2 \frac{\partial^2 u}{\partial y^2} = f \Leftrightarrow \alpha^2 x^2 + \beta^2 y^2 = \gamma^2$

Elliptic curve

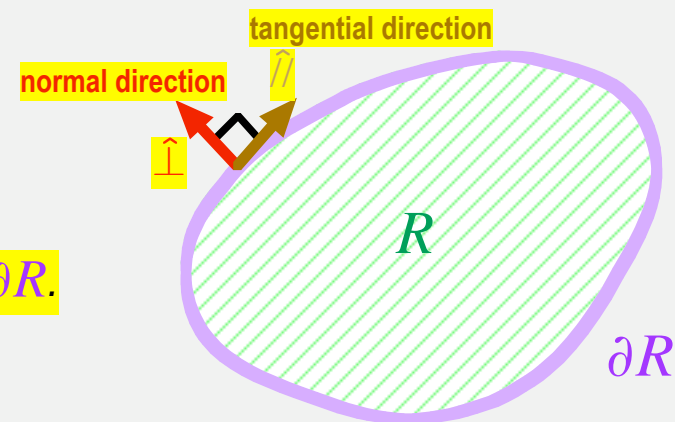
$\Rightarrow$ Poisson or Laplace equation: $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f$ (**time-independent**)

Initial condition & boundary condition

R ... Solution domain

∂R ... Boundary of solution domain

Solution is determined in R
according to the condition given at ∂R .



Example: $u(x, t)$... solution function

Initial condition: $u(x, t_0)$ is given at $t = t_0$.

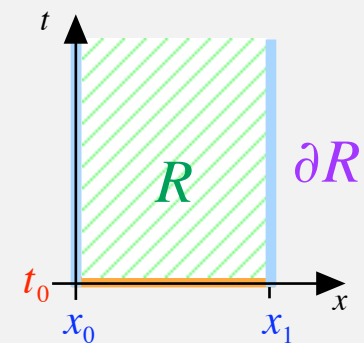
Boundary condition: $u(x_0, t), u(x_1, t)$ are given at $x = x_0$ and x_1 .

Types of conditions (f, g, h ... given functions):

Dirichlet condition... $u = f$ at ∂R

Neumann condition... $\frac{\partial u}{\partial \perp} = g$ or $\frac{\partial u}{\partial //} = h$ at ∂R

Mixed condition... Dirichlet condition + Neumann condition

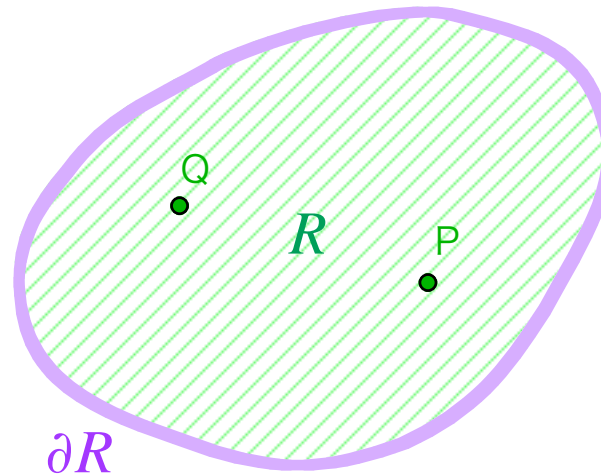


Cartesian coordinates

Well-posedness of a PDE

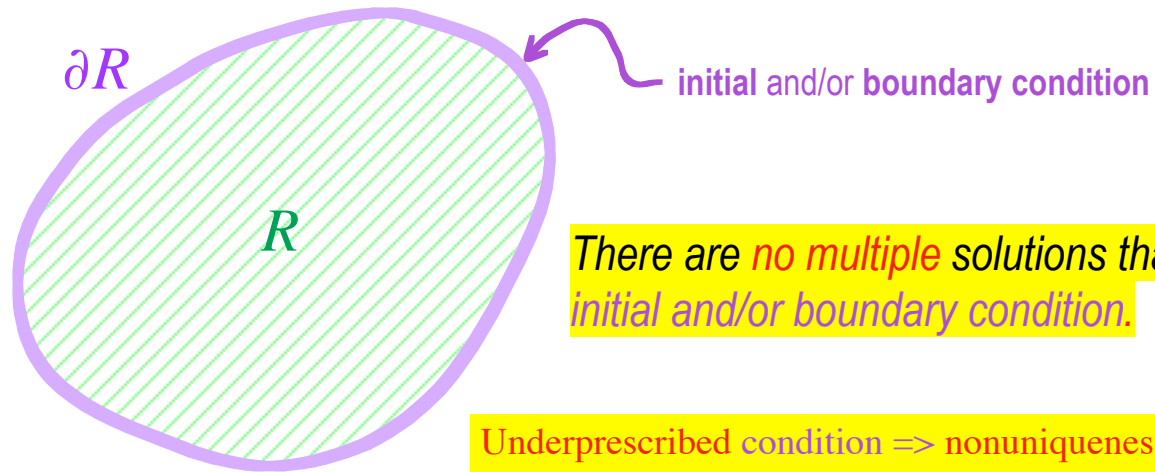
What is well-posedness?

i) Solution must exist.



Solution is determined at **every point** in solution domain (except some singular points such as P and Q).

ii) Solution must be unique.



There are *no multiple* solutions that *satisfy the same initial and/or boundary condition.*

Underprescribed condition $\Rightarrow$ nonuniqueness of solution
Overprescribed condition $\Rightarrow$ unphysical solution

iii) Solution must vary continuously in response to the continuous change of initial and/or boundary condition.

Small change of initial and/or boundary condition always leads to small change of solution.