

# Wave equation

# Wave propagation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

**Time-dependent part... 2nd-order partial derivative**  
**Space-dependent part... 2nd-order partial derivative**

In a three-dimensional case,

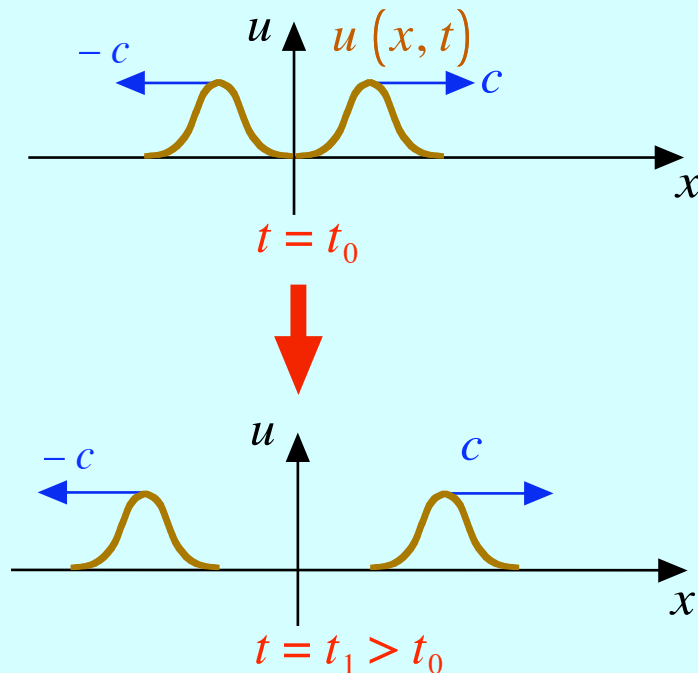
$$\frac{\partial^2 u}{\partial t^2} - c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0$$

## Evolution of a wave profile

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial t} - c \frac{\partial u}{\partial x} = 0$$

$u$ : physical quantity

$c$ : propagation speed



The profile does **not change** with time.  
The profile **moves** with time.

# Linear PDE and Nonlinear PDE

## Linear PDE... linear for solution function and its partial derivatives

Example: **2nd-order linear PDE** (solution function:  $u = u(x, y)$ )

$$A(x, y) \frac{\partial^2 u}{\partial x^2} + B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} + D(x, y) \frac{\partial u}{\partial x} + E(x, y) \frac{\partial u}{\partial y} + F(x, y) u = G(x, y)$$

$G(x, y) = 0 \Rightarrow$  homogeneous  $\longrightarrow$  principle of superposition

$G(x, y) \neq 0 \Rightarrow$  inhomogeneous

**Quasi-linear PDE...** linear for the **highest-order partial derivative** of solution function

$$A, B, C, D, E, F, G \dots \text{depends on } (x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u)$$

e.g.  $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$  This is the 1st-order PDE, so the highest-order is the 1st order.

**Semi-linear PDE...** coefficients of the **highest-order partial derivative** do not depend on solution function

$$A, B, C \dots \text{depends on } (x, y)$$

e.g.  $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0$  (Burgers equation) This is the 2nd-order PDE, so the highest-order is the 2nd order.  
 $\nu$ : constant

## Nonlinear PDE... any PDE except for linear, quasi-linear, semi-linear PDEs

Example: **2nd-order nonlinear PDE** (solution function:  $u = u(x, y)$ )

e.g.  $\left[ \frac{\partial u}{\partial x} \right]^2 + \left[ \frac{\partial u}{\partial y} \right]^2 = u$   $\left[ \frac{\partial u}{\partial x} \right]^2 + \left[ \frac{\partial u}{\partial y} \right]^2 = 1$  (eikonal equation) This is the 1st-order PDE, so the highest-order is the 1st order.

# PDE describing time-dependent processes

Wave propagation, Diffusion, Dispersion

The diagram shows the Burgers equation with color-coded terms and arrows indicating their physical effects:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^3 u}{\partial x^3} = 0$$

- Dispersion** (green box): Points to the third-order derivative term  $\mu \frac{\partial^3 u}{\partial x^3}$ .
- Diffusion** (orange box): Points to the second-order derivative term  $-\nu \frac{\partial^2 u}{\partial x^2}$ .
- Wave propagation (with steepening of a profile... nonlinear effect)** (yellow box): Points to the nonlinear advection term  $u \frac{\partial u}{\partial x}$ .

... Burgers equation

... KdV equation