

Fluid dynamics

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) \dots \text{Partial Differential Equation (PDE)}$$

Integration with respect to t does **NOT** lead to a solution.

$$T(x, z, t) \neq \int \frac{\partial T}{\partial t} dt$$

How can we numerically integrate PDE?



*Introduce **grids** to **space-time** (x, y, z, t)*

Discretization of space-time: introduce grids to space-time ($\Delta x, \Delta y, \Delta z, \Delta t$)

Differential equation

continuous space-time

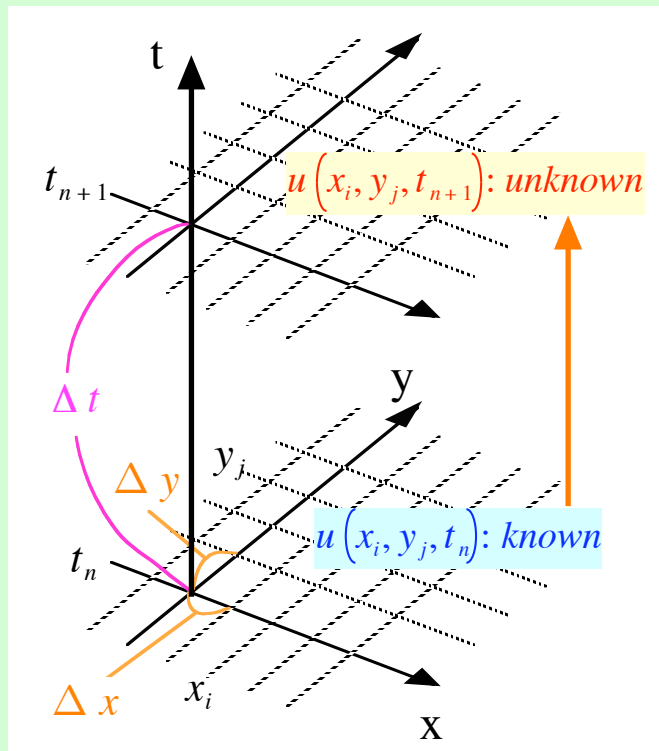
(x, y, z, t)

Difference equation

discrete space-time

$(x_i, y_j, z_k, t_n): i, j, k, n = 1, 2, \dots$

Discretization of 2D space-time
(x, y)



Discretization of 1D space-time
(x)

$$\begin{aligned} \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} &= 0 \\ \Downarrow \\ \frac{\Delta u}{\Delta t} + a \frac{\Delta u}{\Delta x} &= 0 \\ \Downarrow \\ \frac{u(x_i, t_{n+1}) - u(x_i, t_n)}{t_{n+1} - t_n} + a \frac{u(x_{i+1}, t_n) - u(x_i, t_n)}{x_{i+1} - x_i} &= 0 \\ \Downarrow \\ \underbrace{u_i^{n+1}}_{\text{unknown}} &= \underbrace{u_i^n}_{\text{known}} - a \underbrace{\frac{\Delta t}{\Delta x}}_{\text{known}} (\underbrace{u_{i+1}^n}_{\text{known}} - \underbrace{u_i^n}_{\text{known}}) \quad u_i^n = u(x_i, t_n) \\ &\quad (i = 1, 2, \dots) \end{aligned}$$

Computer can derive **unknown values** from **known values** via **four kinds of arithmetic operations** ($+$, $-$, $\times$, $\div$).

Partial differential equations

some basics

0. Typical PDEs used in physics

1. Basic features

- **Three types of PDEs (hyperbolic, parabolic, elliptic)**
- **Initial condition, boundary condition**
- **Characteristics**

2. Hyperbolic PDE

3. Parabolic PDE

4. Elliptic PDE

5. Traditional analytical methods for solving PDEs

Typical PDEs used in physics

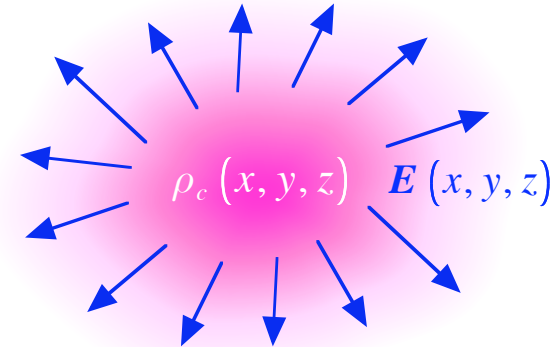
Poisson equation & Laplace equation

Static electric field

$\rho_c(x, y, z)$... charge density



$E(x, y, z)$... static electric field



$$\mathbf{E} = -\nabla\phi_c, \quad \nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon_0}$$

ϵ_0 : permittivity

$\phi_c(x, y, z)$: electrostatic potential



$$\nabla^2 \phi_c = \Delta \phi_c = -\frac{\rho_c}{\epsilon_0} \quad \nabla^2 \equiv \Delta: \text{Laplacian}$$

Poisson equation

Time-dependent part... **does not exist** ($\Rightarrow$ **steady or static**)

Space-dependent part... **2nd-order partial derivative**

In a vacuum case,

$$\nabla^2 \phi_c = \Delta \phi_c = 0$$

Laplace equation

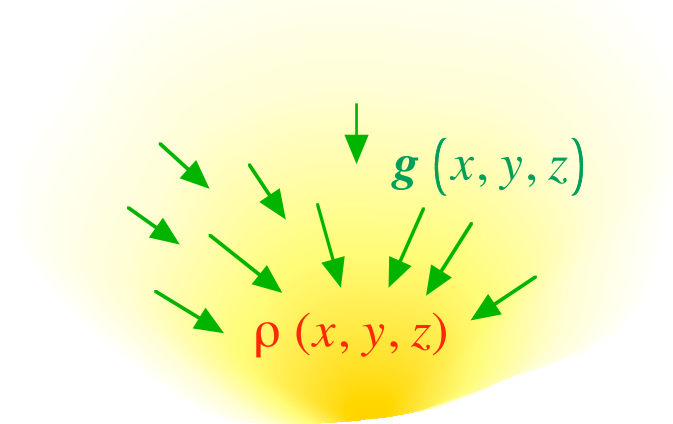
Solution is called **harmonic function**.

Self-gravity field

$\rho(x, y, z)$... mass density



$g(x, y, z)$... self-gravity field



$$g = -\nabla \Phi, \nabla \cdot g = -4\pi G \rho \longrightarrow \nabla^2 \Phi = 4\pi G \rho$$

G : universal gravitational constant

$\Phi(x, y, z)$: gravitational potential

Poisson equation