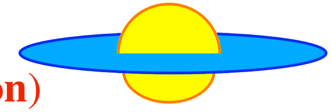
Magnetohydrodynamic model

Weber & Davis 1.5-dimensional model

(depends on *r*, *r & φ–components of a vector* are considered)

(axisymmetric, on the equatorial plane, polytropic, steady, rotation)



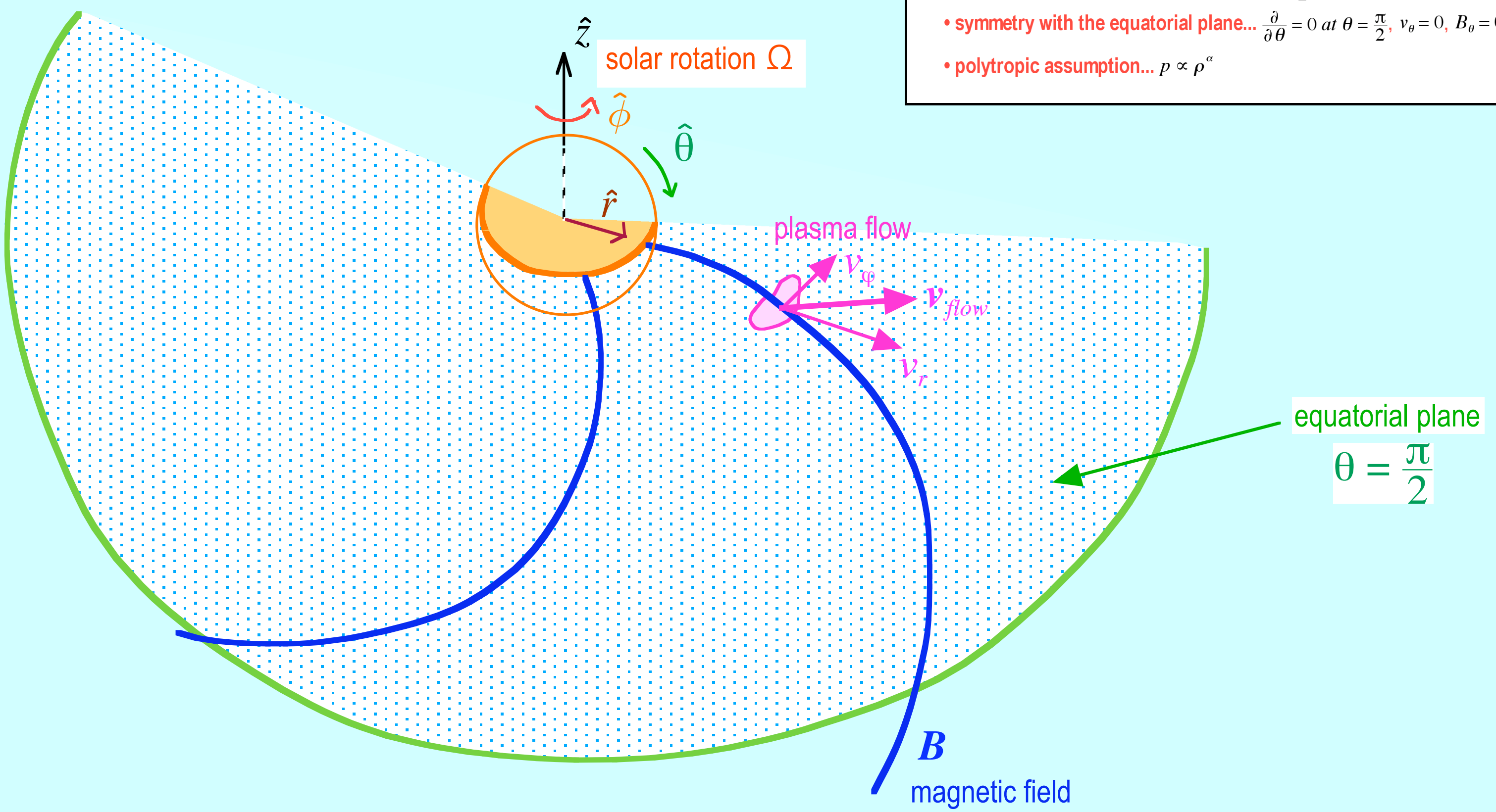
(r, θ, ϕ)... **spherical coordinates with θ = π/2**

$$\boldsymbol{v}(r,\theta,\phi) = v_r(r,\theta,\phi)\,\hat{\boldsymbol{r}} + v_\theta(r,\theta,\phi)\,\hat{\theta} + v_\phi(r,\theta,\phi)\,\hat{\phi}$$

$$\rightarrow v_r\left(r,\theta=\frac{\pi}{2}\right)\hat{\boldsymbol{r}} + v_\phi\left(r,\theta=\frac{\pi}{2}\right)\hat{\theta}$$

Assumption:

- axial symmetry... $\frac{\partial}{\partial \varphi} = 0$
- steady state... $\frac{\partial}{\partial t} = 0$
- only equatorial plane is considered... $\theta = \frac{\pi}{2}$
- symmetry with the equatorial plane... $\frac{\partial}{\partial \theta} = 0 \; at \; \theta = \frac{\pi}{2}$, $v_\theta = 0$, $B_\theta = 0$
- polytropic assumption... $p \propto \rho^\alpha$



Basic equations (differential form): $\frac{\partial}{\partial r} \Rightarrow \frac{d}{dr}$

CGS unit

mass conservation...

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \boldsymbol{v}) = 0 \longrightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 \rho v_r\right) = 0$$

momentum equatoin...

$$\rho \frac{d\boldsymbol{v}}{dt} = -\nabla p + \boldsymbol{j} \times \boldsymbol{B} - \frac{G M_\odot \rho}{r^2}\hat{\boldsymbol{r}} = 0$$

r-component

$$\rho v_r \frac{dv_r}{dr} - \frac{\rho v_\varphi^2}{r} = -\frac{dp}{dr} - \frac{1}{4\pi}\frac{B_\varphi}{r}\frac{d}{dr}\left(r B_\varphi\right) - \frac{G M_\odot \rho}{r^2}$$

ϕ -component

$$\rho v_r \frac{d}{dr}\left(r v_\varphi\right) = \frac{1}{4\pi} B_r \frac{d}{dr}\left(r B_\varphi\right)$$

energy equation...

$$\frac{d}{dt}\left(\frac{p}{\rho^\alpha}\right) = 0 \longrightarrow v_r \frac{d}{dr}\left(\frac{p}{\rho^\alpha}\right) = 0$$

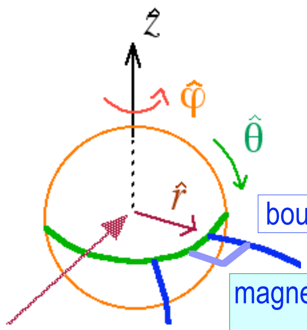
induction equation...

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left(\boldsymbol{v} \times \boldsymbol{B}\right) \xrightarrow{\theta\text{-component}} \frac{d}{dr}\left(r\left[v_\varphi B_r - v_r B_\varphi\right]\right) = 0$$

magnetic flux conservation...

$$\nabla \bullet \boldsymbol{B} = 0 \longrightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2 B_r\right) = 0$$

Basic equations (integral form):



mass conservation... $r^2 \rho v_r = f$ (constant)

energy eqution... $\dfrac{p}{\rho^{\alpha}} = K$ (constant)

induction equation... $v_\varphi B_r - v_r B_\varphi = \dfrac{C_0}{r} = \dfrac{\Omega R_\odot^2 B_0}{r}$

at $r = R_\odot, v_\varphi = \Omega R_\odot, B_r = B_0, B_\varphi = 0 \Rightarrow C_0 = \Omega R_\odot^2 B_0$

Ω... solar rotation rate

(constant)

div B = 0... $r^2 B_r = \Phi = R_\odot^2 B_0$

(constant)

$v_\varphi B_r - v_r B_\varphi = \Omega r B_r$

ϕ -component of momentum $\times$ ***dr*** => angular momentum conservation (per unit mass)

$$\rho v_r \frac{d}{dr}\left(r v_\varphi\right) = \frac{1}{4\pi} B_r \frac{d}{dr}\left(r B_\varphi\right)$$

integrate with r

$$\frac{\rho v_r}{B_r} = \frac{f / r^2}{\Phi / r^2} = \frac{f}{\Phi} = const.$$

$r^2 \rho v_r = f$

$r^2 B_r = \Phi$

$$r v_\varphi - \frac{B_r}{4 \pi \rho v_r} r B_\varphi = L$$

... angular momentum per unit mass (constant)

$$L = v_\varphi\left(r = R_\odot\right) R_\odot = \Omega R_\odot R_\odot$$

Angular momentum conservation (per unit mass)

r-component of momentum • dr => energy conservation (per unit mass)

$$v_r \frac{dv_r}{dr} - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{1}{4\pi}\frac{B_\varphi}{\rho\, r}\frac{d}{dr}(r B_\varphi) - \frac{G M_\odot}{r^2}$$

$$\frac{p}{\rho^\alpha} = K$$

$$\frac{\alpha}{\alpha-1}\frac{d}{dr}\left(\frac{p}{\rho}\right)$$

$$\rho\, v_r \frac{d}{dr}(r\, v_\varphi) = \frac{1}{4\pi} B_r \frac{d}{dr}(r\, B_\varphi)$$

ϕ -component of momentum eq.

$$\frac{v_r}{r}\frac{B_\varphi}{B_r}\frac{d}{dr}(r\, v_\varphi)$$

$$v_\varphi B_r - v_r B_\varphi = \Omega\, r B_r \Rightarrow \frac{B_\varphi}{B_r} = \frac{v_\varphi - r\,\Omega}{v_r}$$

Induction eq.

$$\frac{v_\varphi - r\,\Omega}{r}\frac{d}{dr}(r\, v_\varphi) = (v_\varphi - r\,\Omega)\frac{d}{dr}(v_\varphi - r\,\Omega) + \frac{v_\varphi^2}{r} - r\,\Omega^2$$

$$v_r \frac{dv_r}{dr} = -\frac{\alpha}{\alpha-1}\frac{d}{dr}\left(\frac{p}{\rho}\right) - (v_\varphi - r\,\Omega)\frac{d}{dr}(v_\varphi - r\,\Omega) - \frac{G M_\odot}{r^2} + r\,\Omega^2$$

integrate with r

Energy conservation (per unit mass)

radial flow kinetic energy, rotational energy, thermal energy, gravitational energy, centrifugal energy

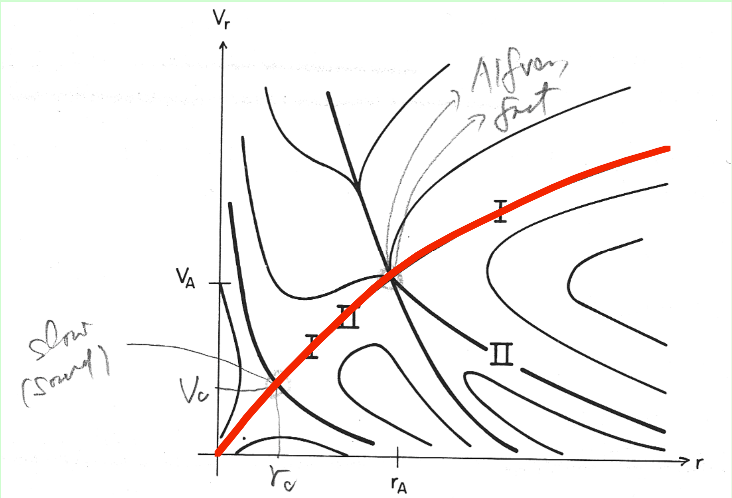
$$\frac{1}{2}v_r^2 + \frac{1}{2}(v_\varphi - r\,\Omega)^2 + \frac{\alpha}{\alpha-1}\frac{p}{\rho} - \frac{G M_\odot}{r} - \frac{1}{2}(r\,\Omega)^2 = E$$

... energy per unit mass (constant)

6 equations for 6 quantities $(\rho, v_r, v_\varphi, p, B_r, B_\varphi)$

→ **one equation for one quantity** $(v_r(r))$

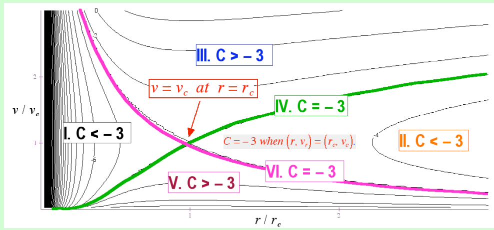
Profile of $v_r\,(r)$



There are three critical points corresponding to three characteristic wave speeds: slow-mode speed, Alfvén speed, fast-mode speed.

Solution I is appropriate for solar wind.

$$r_c \sim 5\,R_\odot\ ,\ \ r_A \sim 10\,R_\odot\ \ with\ T \sim 10^6\ K$$



Parker's solution (there is only one critical point corresponding to sound speed)

For details, see http://163.180.179.74/~magara/page31/Topics/SolarWind/sw.html.