Physical properties of shear Alfvén wave (SAW): (obtained from amplitude relations)

Four vector relation $(\boldsymbol{k}, \boldsymbol{B}_0, \boldsymbol{v}^*, \boldsymbol{B}^*)$

- $\boldsymbol{k} \bullet \boldsymbol{B}^* = 0 \; (\nabla \bullet \boldsymbol{B}_1 = 0)$ => $\boldsymbol{k} \perp \boldsymbol{B}^*$, $\boldsymbol{k} \bullet \boldsymbol{v}^* = 0 \; (\nabla \bullet \boldsymbol{v}_1 = 0)$ => $\boldsymbol{k} \perp \boldsymbol{v}^*$ ... **transverse wave (incompressive)**
- $\omega \boldsymbol{B}^* = -\boldsymbol{k} \times [\boldsymbol{v}^* \times \boldsymbol{B}_0] = -[(\boldsymbol{k} \bullet \boldsymbol{B}_0)\boldsymbol{v}^* - (\boldsymbol{k} \bullet \boldsymbol{v}^*)\boldsymbol{B}_0] = -(\boldsymbol{k} \bullet \boldsymbol{B}_0)\boldsymbol{v}^*$

  $(\boldsymbol{k} \bullet \boldsymbol{v}^*) = 0$

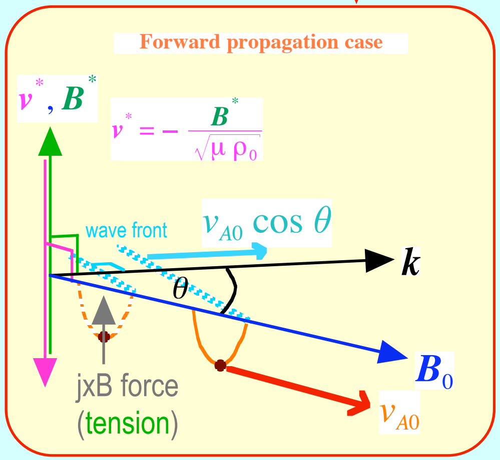
$$\boldsymbol{v}^* = -\frac{\omega}{(\boldsymbol{k} \bullet \boldsymbol{B}_0)} \boldsymbol{B}^* \Rightarrow \begin{cases} \text{phase difference of } \pi \text{ for } \boldsymbol{k} \bullet \boldsymbol{B}_0 > 0 \text{ (forward propagation)} \\ \text{same phase for } \boldsymbol{k} \bullet \boldsymbol{B}_0 < 0 \text{ (backward propagation)} \end{cases}$$

Oscillation directions of **velocity** and **magnetic field** are **aligned:** $\boldsymbol{v}^* = \begin{smallmatrix} - \text{(forward propagation)} \\ + \text{(backward propagation)} \end{smallmatrix} \frac{\boldsymbol{B}^*}{\sqrt{\mu \rho_0}}$

They are **perpendicular to B_0.**

- $\boxed{\boldsymbol{v}^* \bullet \boldsymbol{B}_0 = 0}$, $\boldsymbol{v}^* = \mp \frac{\boldsymbol{B}^*}{\sqrt{\mu_0 \rho_0}} \Rightarrow \boxed{\boldsymbol{B}^* \bullet \boldsymbol{B}_0 = 0}$

  ... common property



- **Driving force:**

$$\boldsymbol{j}^* \times \boldsymbol{B}_0 = \frac{1}{\mu_0}[\boldsymbol{k} \times \boldsymbol{B}^*] \times \boldsymbol{B}_0 = \boxed{(\boldsymbol{B}_0 \bullet \boldsymbol{k}) \frac{\boldsymbol{B}^*}{\mu_0}} - \boxed{\left(\frac{\boldsymbol{B}^* \bullet \boldsymbol{B}_0}{\mu_0}\right)} \boldsymbol{k}$$

magnetic tension: $\neq 0$ except $\theta = \pi/2$

magnetic pressure: $= 0$

Magnetic tension is the driving force of SAW. $\quad p_m^* = \frac{2\boldsymbol{B}^* \bullet \boldsymbol{B}_0}{2\mu_0}$

2) $\boldsymbol{k} \bullet \boldsymbol{v}^* \neq 0 \Rightarrow \omega^2 - k^2 v_{A0}^2 = 0$ ... *similar to sound wave* **(compressive)**

=> **Compressional Alfvén wave**

Velocity diagram

Dispersion relation of compressional Alfvén wave: $\omega(\boldsymbol{k}) = \pm k\, v_{A0}$

Phase velocity

$$\boldsymbol{v}_p = \frac{\omega}{k}\hat{\boldsymbol{k}} = \pm v_{A0}\hat{\boldsymbol{k}}$$

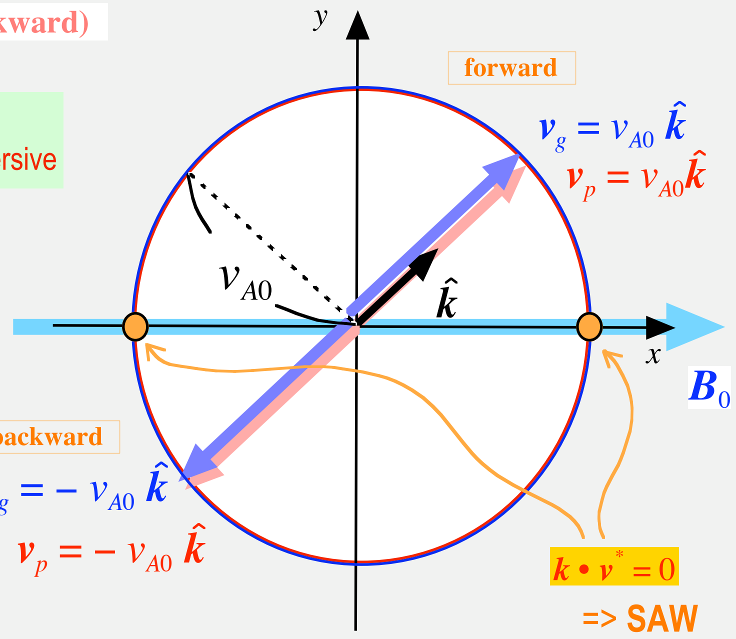
(+... forward, −... backward)

$|\boldsymbol{v}_p|$ is independent of the direction of **k** ($\hat{k}$)... isotropic
$|\boldsymbol{v}_p|$ is independent of the magnitude of **k** ($|k|$)... non-dispersive

Group velocity

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}} = \pm v_{A0}\hat{\boldsymbol{k}}$$

Same speed in all directions
Equal to the phase velocity



Physical properties of compressional Alfvén wave (CAW): (obtained from amplitude relations)

Four vector relation $(\boldsymbol{k}, \boldsymbol{B}_0, \boldsymbol{v}^*, \boldsymbol{B}^*)$

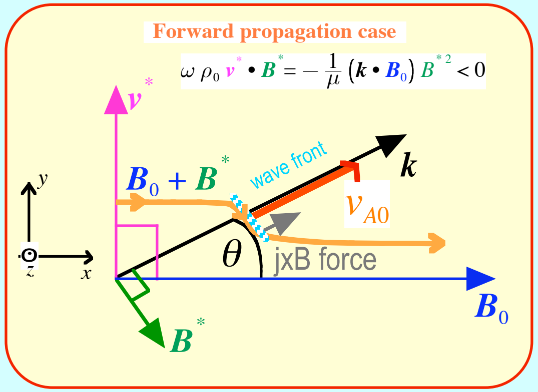
- $\boldsymbol{k} \bullet \boldsymbol{B}^* = 0 \Rightarrow \boldsymbol{k} \perp \boldsymbol{B}^*$ (div B = 0), $\boldsymbol{v}^* \bullet \boldsymbol{B}_0 = 0 \Rightarrow \boldsymbol{v}^* \perp \boldsymbol{B}_0$ ... common property
- $$\omega \rho_0 \boldsymbol{v}^* = -\frac{1}{\mu_0}\left[\boldsymbol{k} \times \boldsymbol{B}^*\right] \times \boldsymbol{B}_0 = -\frac{1}{\mu_0}\left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)\boldsymbol{B}^* - \underbrace{\left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0\right)}_{\neq 0}\boldsymbol{k}\right]$$

  => $\boldsymbol{v}^*$ *is on the plane formed by* $\boldsymbol{B}^*$ *and* $\boldsymbol{k}$ $(\boldsymbol{k} \perp \boldsymbol{B}^*)$
- $$\omega \boldsymbol{B}^* = -\boldsymbol{k} \times \left[\boldsymbol{v}^* \times \boldsymbol{B}_0\right] = -\left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)\boldsymbol{v}^* - \underbrace{\left(\boldsymbol{k} \bullet \boldsymbol{v}^*\right)}_{\neq 0}\boldsymbol{B}_0\right]$$

  $\boldsymbol{v}^*$ and $\boldsymbol{B}^*$ are not parallel

  => $\boldsymbol{B}^*$ *is on the plane formed by* $\boldsymbol{v}^*$ *and* $\boldsymbol{B}_0$ $(\boldsymbol{v}^* \perp \boldsymbol{B}_0)$

Four vectors $\boldsymbol{v}^*$, $\boldsymbol{B}^*$, $\boldsymbol{k}$, $\boldsymbol{B}_0$ are on the same plane.



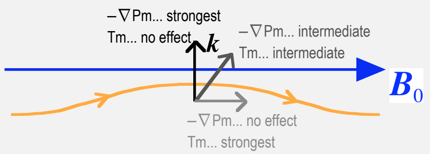
- **Driving force:**

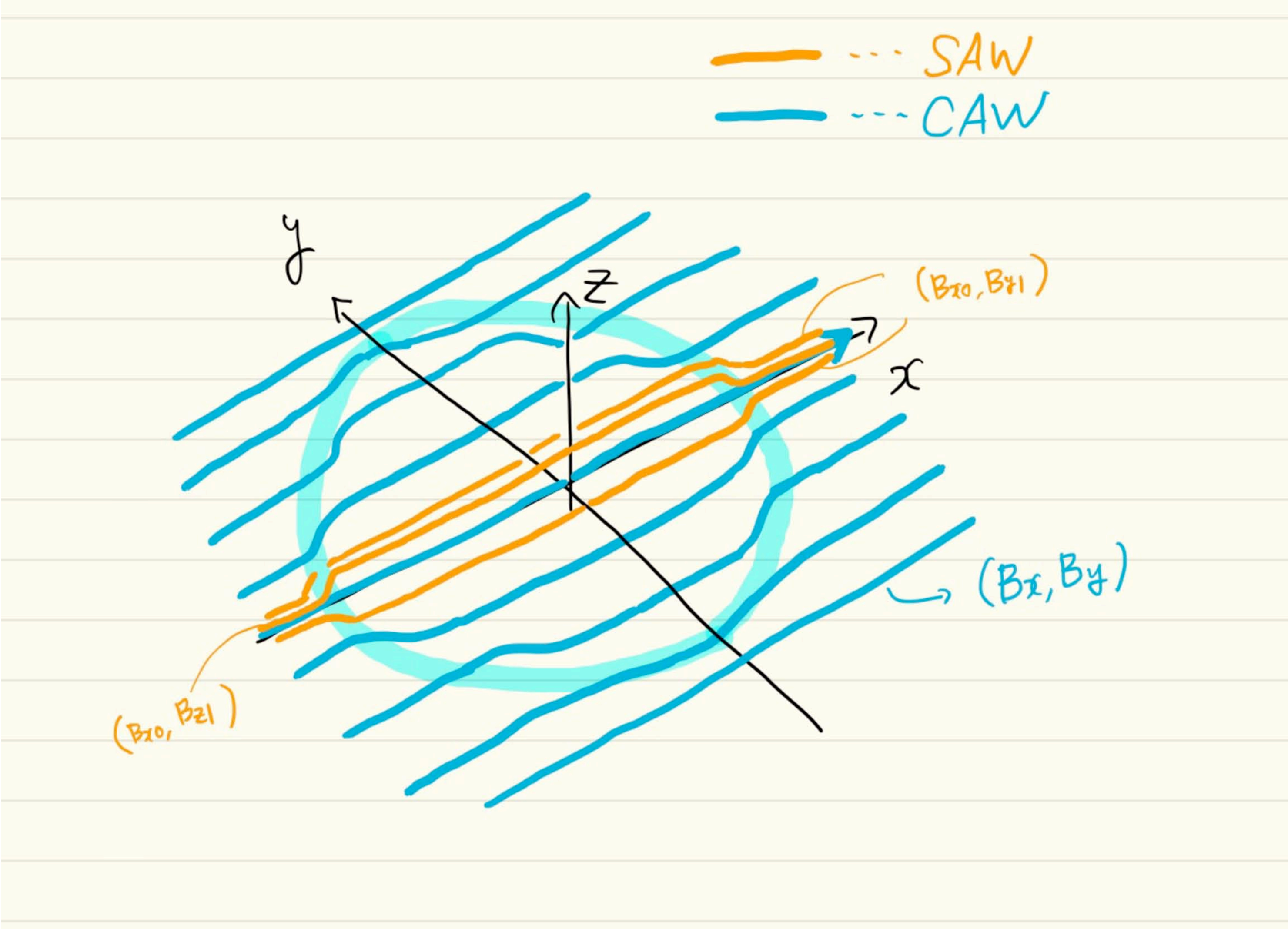
$$\boldsymbol{j}^* \times \boldsymbol{B}_0 = \frac{1}{\mu_0}\left[\boldsymbol{k} \times \boldsymbol{B}^*\right] \times \boldsymbol{B}_0 = \underbrace{\left(\boldsymbol{B}_0 \bullet \boldsymbol{k}\right)}_{\text{magnetic tension}}\frac{\boldsymbol{B}^*}{\mu_0} - \underbrace{\left(\frac{\boldsymbol{B}^* \bullet \boldsymbol{B}_0}{\mu_0}\right)}_{\text{magnetic pressure}}\boldsymbol{k}$$

magnetic tension: $\neq 0$ except $\theta = \pm\pi/2$

magnetic pressure: $\neq 0$ except $\theta = 0, \pi$

Both **tension force** and **magnetic pressure gradient force** are the driving force of CAW.




SAW
CAW
y
z
x
(Bx0, By1)
(Bx, By)
(Bx0, Bz1)

Magnetoacoustic wave

$p_0 \neq 0$, $p_1 \neq 0, B_0 \neq 0, B_1 \neq 0$

Linearized ideal MHD equations

$$
\begin{cases}
\dfrac{\partial \rho_1 (x,y,z,t)}{\partial t} = -\rho_0 \nabla \bullet \boldsymbol{v}_1 (x,y,z,t) \\
\rho_0 \dfrac{\partial \boldsymbol{v}_1 (x,y,z,t)}{\partial t} = -\nabla p_1 (x,y,z,t) + \dfrac{1}{\mu_0} \left(\nabla \times \boldsymbol{B}_1 (x,y,z,t)\right) \times \boldsymbol{B}_0 \\
\dfrac{\partial p_1 (x,y,z,t)}{\partial t} = \gamma \dfrac{p_0}{\rho_0} \dfrac{\partial \rho_1 (x,y,z,t)}{\partial t} \\
\dfrac{\partial \boldsymbol{B}_1 (x,y,z,t)}{\partial t} = \nabla \times \left(\boldsymbol{v}_1 (x,y,z,t) \times \boldsymbol{B}_0\right)
\end{cases}
$$

Partial differential equations

$$\rho_1 (x,y,z,t) = \rho^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$$

$$\boldsymbol{v}_1 (x,y,z,t) = \boldsymbol{v}^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$$

$$p_1 (x,y,z,t) = p^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$$

$$\boldsymbol{B}_1 (x,y,z,t) = \boldsymbol{B}^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$$

$$-i\omega \rho^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} = -\rho_0 i \boldsymbol{k} \bullet \boldsymbol{v}^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$$

$$-i\omega \rho_0 \boldsymbol{v}^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} = -i \boldsymbol{k} p^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} + \frac{1}{\mu_0} \left[i \boldsymbol{k} \times \boldsymbol{B}^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} \right] \times \boldsymbol{B}_0$$

$$-i\omega p^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} = -\gamma \frac{p_0}{\rho_0} i \omega \rho^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$$

$$-i\omega \boldsymbol{B}^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} = i \boldsymbol{k} \times \left[\boldsymbol{v}^* e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)} \times \boldsymbol{B}_0 \right]$$

$$\omega \rho^* = \rho_0 \boldsymbol{k} \bullet \boldsymbol{v}^*$$

$$\omega \rho_0 \boldsymbol{v}^* = \boldsymbol{k} p^* - \frac{1}{\mu_0} \left(\boldsymbol{k} \times \boldsymbol{B}^* \right) \times \boldsymbol{B}_0$$

$$\omega p^* = \gamma \frac{p_0}{\rho_0} \omega \rho^*$$

$$\omega \boldsymbol{B}^* = -\boldsymbol{k} \times \left(\boldsymbol{v}^* \times \boldsymbol{B}_0 \right)$$

Algebraic equations

$$\omega \rho^* = \rho_0 \boldsymbol{k} \bullet \boldsymbol{v}^*$$

$$\omega \rho_0 \boldsymbol{v}^* = \boldsymbol{k} p^* - \frac{1}{\mu_0} \left(\boldsymbol{k} \times \boldsymbol{B}^* \right) \times \boldsymbol{B_0} = = \boldsymbol{k} p^* - \frac{1}{\mu_0} \left[\left(\boldsymbol{k} \bullet \boldsymbol{B_0} \right) \boldsymbol{B}^* - \left(\boldsymbol{B}^* \bullet \boldsymbol{B_0} \right) \boldsymbol{k} \right]$$

$$\omega p^* = \gamma \frac{p_0}{\rho_0} \omega \rho^*$$

$$\omega \boldsymbol{B}^* = - \boldsymbol{k} \times \left(\boldsymbol{v}^* \times \boldsymbol{B_0} \right) = - \left[\left(\boldsymbol{k} \bullet \boldsymbol{B_0} \right) \boldsymbol{v}^* - \left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right) \boldsymbol{B_0} \right]$$

$$\nabla \bullet \boldsymbol{B_1} = 0 \rightarrow \boldsymbol{k} \bullet \boldsymbol{B}^* = 0$$

oscillation direction of magnetic field ⊥ propagation direction

$\bullet \boldsymbol{B}_0$

$$\left(\omega \rho_0 \boldsymbol{v}^* \right) \bullet \boldsymbol{B}_0 = \left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right) p^* - \left(\frac{1}{\mu_0} \left[\boldsymbol{k} \times \boldsymbol{B}^* \right] \times \boldsymbol{B_0} \right) \bullet \boldsymbol{B}_0 \rightarrow \omega \rho_0 \left(\boldsymbol{v}^* \bullet \boldsymbol{B}_0 \right) = \left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right) p^*$$

$= 0$

$\bullet \boldsymbol{k}$

$$\left(\omega \rho_0 \boldsymbol{v}^* \right) \bullet \boldsymbol{k} = \left(\boldsymbol{k} \bullet \boldsymbol{k} \right) p^* - \left(\frac{1}{\mu_0} \left[\left(\boldsymbol{k} \bullet \boldsymbol{B_0} \right) \boldsymbol{B}^* - \left(\boldsymbol{B}^* \bullet \boldsymbol{B_0} \right) \boldsymbol{k} \right] \right) \bullet \boldsymbol{k}$$

$$\Rightarrow \omega \rho_0 \left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right) = k^2 p^* - \frac{1}{\mu_0} \left[\left(\boldsymbol{k} \bullet \boldsymbol{B_0} \right) \left(\boldsymbol{k} \bullet \boldsymbol{B}^* \right) - \left(\boldsymbol{B}^* \bullet \boldsymbol{B_0} \right) k^2 \right]$$

$= 0$

$\bullet \boldsymbol{B}_0$

$$\omega \boldsymbol{B}^* \bullet \boldsymbol{B}_0 = - \left[\left(\boldsymbol{k} \bullet \boldsymbol{B_0} \right) \boldsymbol{v}^* - \left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right) \boldsymbol{B_0} \right] \bullet \boldsymbol{B}_0$$

$$\Rightarrow \omega \boldsymbol{B}^* \bullet \boldsymbol{B}_0 = - \left[\left(\boldsymbol{k} \bullet \boldsymbol{B_0} \right) \boldsymbol{v}^* \bullet \boldsymbol{B}_0 - \left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right) B_{\boldsymbol{0}}^2 \right] \rightarrow \omega \boldsymbol{B}^* \bullet \boldsymbol{B}_0 = - \left[\left(\boldsymbol{k} \bullet \boldsymbol{B_0} \right)^2 \frac{p^*}{\omega \rho_0} - \left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right) B_{\boldsymbol{0}}^2 \right]$$

$$\boldsymbol{v}^* \bullet \boldsymbol{B}_0 = \frac{\boldsymbol{k} \bullet \boldsymbol{B}_0}{\omega \rho_0} p^*$$

$\bullet \boldsymbol{k}$

$0 = 0$

$$\omega \rho^* = \rho_0 \, \boldsymbol{k} \bullet \boldsymbol{v}^* \Rightarrow \rho^* = \frac{\rho_0}{\omega} \boldsymbol{k} \bullet \boldsymbol{v}^*$$

$$\omega p^* = \gamma \frac{p_0}{\rho_0} \omega \rho^* \Rightarrow p^* = \gamma \frac{p_0}{\rho_0} \rho^*$$

$$p^* = \gamma \frac{p_0}{\omega} \boldsymbol{k} \bullet \boldsymbol{v}^*$$

gas pressure vs. magnetic pressure

$$\left(\frac{\omega^2}{\gamma \frac{p_0}{\rho_0}} - k^2 \right) p^* = \frac{1}{\mu_0} \left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0 \right) k^2$$

$$p_m^* = \frac{2 \, \boldsymbol{B}^* \bullet \boldsymbol{B}_0}{2 \mu_0}$$

$$\omega \rho_0 \left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right) = k^2 p^* + \frac{1}{\mu_0} \left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0 \right) k^2$$

$$\omega \boldsymbol{B}^* \bullet \boldsymbol{B}_0 = - \left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right)^2 \frac{p^*}{\omega \rho_0} - \left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right) B_0^2 \right]$$

eliminate $\boldsymbol{B}^* \bullet \boldsymbol{B}_0$

$$\left(\omega^2 - \frac{B_0^2}{\mu_0 \rho_0} k^2 \right) \left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right) = \left[\omega k^2 - \frac{k^2}{\omega} \frac{\left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right)^2}{\mu_0 \rho_0} \right] \frac{p^*}{\rho_0}$$

eliminate p^*

$$\left[\omega^4 - \left(\gamma \frac{p_0}{\rho_0} + \frac{B_0^2}{\mu_0 \rho_0} \right) k^2 \omega^2 + \gamma \frac{p_0}{\rho_0} \frac{\left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right)^2}{\mu_0 \rho_0} k^2 \right] \left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right) = 0$$

$$\left[\omega^4 - \left(\gamma \frac{p_0}{\rho_0} + \frac{B_0^2}{\mu_0 \rho_0}\right) k^2 \omega^2 + \gamma \frac{p_0}{\rho_0} \frac{(\boldsymbol{k} \bullet \boldsymbol{B}_0)^2}{\mu_0 \rho_0} k^2\right] (\boldsymbol{k} \bullet \boldsymbol{v}^*) = 0$$

1) $\boldsymbol{k} \bullet \boldsymbol{v}^* = 0$**... incompressive**

2) $\boldsymbol{k} \bullet \boldsymbol{v}^* \neq 0$**... compressive**

1) $\boldsymbol{k} \bullet \boldsymbol{v}^* = 0 \Rightarrow \nabla \bullet \boldsymbol{v}_1 = 0$ ... *incompressional wave* => **Shear Alfvén wave**

$$\omega \rho^* = \rho_0 \boldsymbol{k} \bullet \boldsymbol{v}^*$$

$$\omega \rho_0 \boldsymbol{v}^* = \boldsymbol{k} p^* - \frac{1}{\mu_0} \left(\boldsymbol{k} \times \boldsymbol{B}^* \right) \times \boldsymbol{B}_0 = = \boldsymbol{k} p^* - \frac{1}{\mu_0} \left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right) \boldsymbol{B}^* - \left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0 \right) \boldsymbol{k} \right]$$

$$\omega p^* = \gamma \frac{p_0}{\rho_0} \omega \rho^*$$

$$\omega \boldsymbol{B}^* = - \boldsymbol{k} \times \left(\boldsymbol{v}^* \times \boldsymbol{B}_0 \right) = - \left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right) \boldsymbol{v}^* - \underbrace{\left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right)}_{= 0} \boldsymbol{B}_0 \right]$$

$\bullet \boldsymbol{B}_0$

$$\boldsymbol{v}^* \bullet \boldsymbol{B}_0 = \frac{\boldsymbol{k} \bullet \boldsymbol{B}_0}{\omega \rho_0} p^* = 0$$

$$\rho^* = 0$$

$$p^* = \gamma \frac{p_0}{\rho_0} \rho^* = 0$$

$$\omega \rho_0 \boldsymbol{v}^* = - \frac{1}{\mu_0} \left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right) \boldsymbol{B}^* - \left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0 \right) \boldsymbol{k} \right]$$

$$\omega \boldsymbol{B}^* = - \left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right) \boldsymbol{v}^*$$

eliminate $\boldsymbol{B}^*$

$$\omega^2 \rho_0 \boldsymbol{v}^* = - \frac{1}{\mu_0} \left[- \left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right)^2 \boldsymbol{v}^* + \underbrace{\left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right)}_{= 0} \left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right) \boldsymbol{B}_0 + \left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right) \underbrace{\left(\boldsymbol{v}^* \bullet \boldsymbol{B}_0 \right)}_{= 0} - \underbrace{\left(\boldsymbol{k} \bullet \boldsymbol{v}^* \right)}_{= 0} B_0^2 \right] \boldsymbol{k} \right]$$

$$\left[\omega^2 - \frac{\left(\boldsymbol{k} \bullet \boldsymbol{B}_0 \right)^2}{\mu_0 \rho_0} \right] \boldsymbol{v}^* = \boldsymbol{0} \Rightarrow \omega = \pm \frac{\boldsymbol{k} \bullet \boldsymbol{B}_0}{\sqrt{\mu_0 \rho_0}}$$ **... dispersion relation of SAW**

Velocity diagram

Dispersion relation of SAW: $\omega(\mathbf{k}) = \pm \frac{\mathbf{k} \bullet \mathbf{B}_0}{\sqrt{\mu_0 \rho_0}} = \pm k\, v_{A0} \cos\theta$

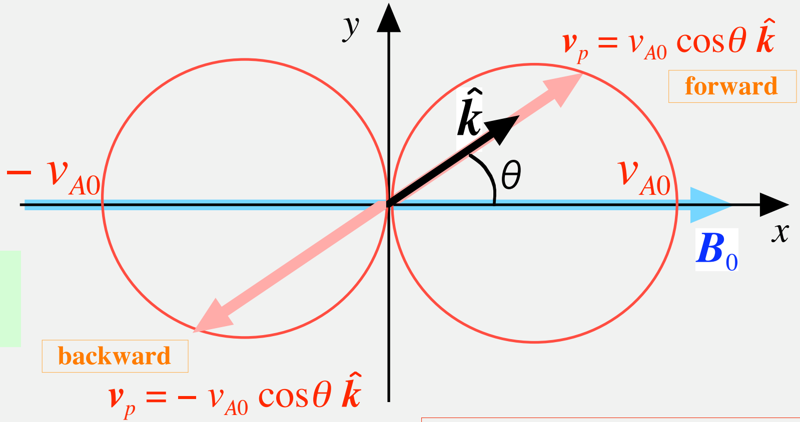
Phase velocity

$$\mathbf{v}_p = \frac{\omega}{k}\hat{\mathbf{k}} = \pm v_{A0} \cos\theta\, \hat{\mathbf{k}}$$

(+... forward, −... backward)

$|\mathbf{v}_p|$ is dependent on the direction of **k** ($\hat{k}$)... non-isotropic

$|\mathbf{v}_p|$ is independent of the magnitude of **k** ($|k|$)... non-dispersive

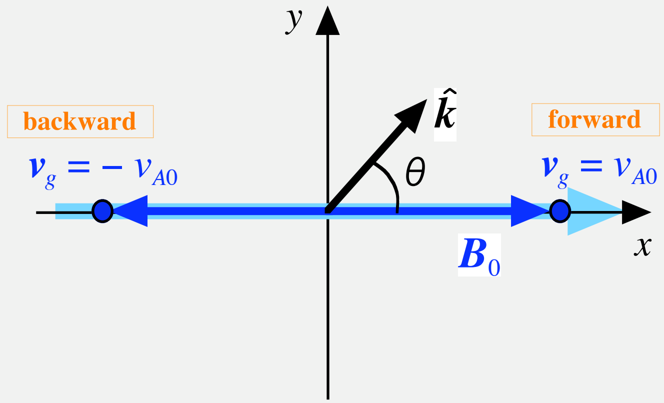


Consider why shear Alfvén wave is not affected by gas pressure.

Group velocity

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}} = \pm \begin{pmatrix} \frac{\partial}{\partial k_x} \\ \frac{\partial}{\partial k_y} \\ \frac{\partial}{\partial k_z} \end{pmatrix} \frac{\mathbf{k} \bullet \mathbf{B}_0}{\sqrt{\mu_0 \rho_0}}$$

$$= \pm \begin{pmatrix} \frac{\partial}{\partial k_x} \\ \frac{\partial}{\partial k_y} \\ \frac{\partial}{\partial k_z} \end{pmatrix} \frac{k_x B_0 + k_y 0 + k_z 0}{\sqrt{\mu_0 \rho_0}} = \pm \begin{pmatrix} \frac{B_0}{\sqrt{\mu_0 \rho_0}} \\ 0 \\ 0 \end{pmatrix} = \pm \begin{pmatrix} v_{A0} \\ 0 \\ 0 \end{pmatrix}$$



Energy (wave packet) is always transported along magnetic field B_0.

2) $\boldsymbol{k} \bullet \boldsymbol{v}^* \neq 0 \Rightarrow \omega^4 - \left(\gamma \frac{p_0}{\rho_0} + \frac{B_0^2}{\mu_0 \rho_0}\right) k^2 \omega^2 + \gamma \frac{p_0}{\rho_0} \frac{(\boldsymbol{k} \bullet \boldsymbol{B_0})^2}{\mu_0 \rho_0} k^2 = 0$

compressional wave

$$\omega^4 - \left(c_{S0}^2 + v_{A0}^2\right) k^2 \omega^2 + c_{S0}^2 v_{A0}^2 k^4 \cos^2\theta = 0$$

compressional Alfvén wave => shear Alfvén wave

When θ is zero (propagation along magnetic field), this is reduced to

$$\left(\omega^2 - k^2 c_{S0}^2\right)\left(\omega^2 - k^2 v_{A0}^2\right) = 0,$$

so sound wave and shear Alfvén wave are completely decoupled.

Consider this from a viewpoint of driving force.

Two types of compressional MHD wave: **slow wave** & **fast wave**

$$\omega(\boldsymbol{k}) = \pm k \sqrt{\frac{c_{S0}^2 + v_{A0}^2}{2} \pm \frac{\sqrt{c_{S0}^4 + v_{A0}^4 - 2 c_{S0}^2 v_{A0}^2 \cos 2\theta}}{2}}$$

$\cos 2\theta = 2\cos^2\theta - 1$

First $\pm$: forward (+), backward (–)

Second $\pm$: fast wave (+), slow wave (–)

Slow wave

Dispersion relation: $\omega^2 = k^2 \left(\frac{c_{S0}^2 + v_{A0}^2}{2} - \frac{\sqrt{c_{S0}^4 + v_{A0}^4 - 2\, c_{S0}^2 v_{A0}^2 \cos 2\theta}}{2} \right) \le k^2 c_{S0}^2$ Confirm this.

Gas pressure and magnetic pressure work **in an uncooperative way**.

gas pressure vs. magnetic pressure

$$\left(\frac{\omega^2}{\gamma \frac{p_0}{\rho_0}} - k^2 \right) p^* = \frac{1}{\mu_0} \left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0 \right) k^2$$

$\frac{\omega^2}{c_{s0}^2} - k^2 \le 0$ → **out of phase**

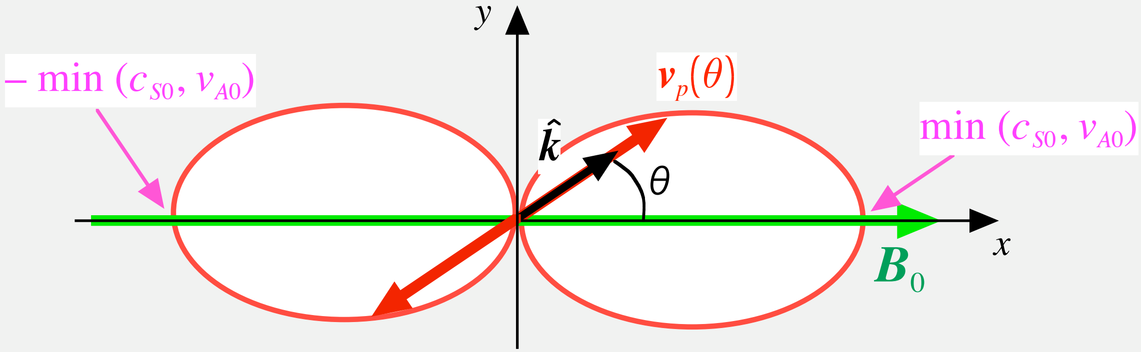
Phase velocity

$$\boldsymbol{v}_p = \frac{\omega(k,\theta)}{k} \hat{\boldsymbol{k}} = \pm \sqrt{\frac{c_{S0}^2 + v_{A0}^2}{2} - \frac{\sqrt{c_{S0}^4 + v_{A0}^4 - 2\, c_{S0}^2 v_{A0}^2 \cos 2\theta}}{2}}\ \hat{\boldsymbol{k}} = \boldsymbol{v}_p(\theta)$$

only depends on the direction of $\boldsymbol{k}$

$|\mathbf{v}_p|$ is dependent on the direction of **k** ($\hat{\boldsymbol{k}}$)... non-isotropic

$|\mathbf{v}_p|$ is independent of the magnitude of **k** ($|\boldsymbol{k}|$)... non-dispersive



Group velocity

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}}\,,\ \omega\left(k,\theta\right) = \pm\, k \sqrt{\frac{c_{S0}^2 + v_{A0}^2}{2} - \frac{\sqrt{c_{S0}^4 + v_{A0}^4 - 2\, c_{S0}^2\, v_{A0}^2 \cos 2\theta}}{2}}$$

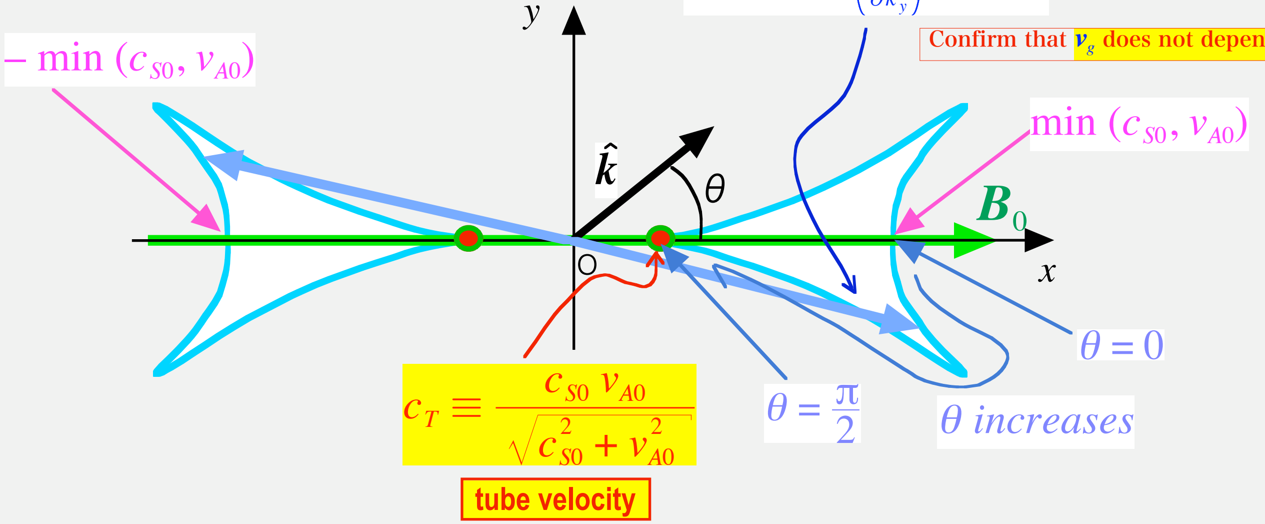
$$k = \sqrt{k_x^2 + k_y^2}$$

$$\cos\left(2\theta\right) = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - \frac{k_y^2}{k_x^2}}{1 + \frac{k_y^2}{k_x^2}} = \frac{k_x^2 - k_y^2}{k_x^2 + k_y^2}$$

$$\omega\left(k_x, k_y\right)$$

$$\boldsymbol{v}_g\left(k,\theta\right) = \begin{pmatrix} \frac{\partial \omega}{\partial k_x} \\ \frac{\partial \omega}{\partial k_y} \end{pmatrix} = \boldsymbol{v}_g\left(\theta\right)$$

Confirm that $\boldsymbol{v}_g$ does not depend on $k = |\boldsymbol{k}|$.



Fast wave

Dispersion relation: $\omega^2 = k^2\left(\frac{c_{S0}^2 + v_{A0}^2}{2} + \frac{\sqrt{c_{S0}^4 + v_{A0}^4 - 2\,c_{S0}^2 v_{A0}^2 \cos 2\theta}}{2}\right) \geq k^2 c_{S0}^2$ Confirm this.

Gas pressure and magnetic pressure work **in a cooperative way**.

gas pressure vs. magnetic pressure

$$\left(\frac{\omega^2}{\gamma \frac{p_0}{\rho_0}} - k^2\right) p^* = \frac{1}{\mu_0}\left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0\right) k^2$$

$$\frac{\omega^2}{c_{s0}^2} - k^2 \geq 0$$

→ **in phase**

Phase velocity

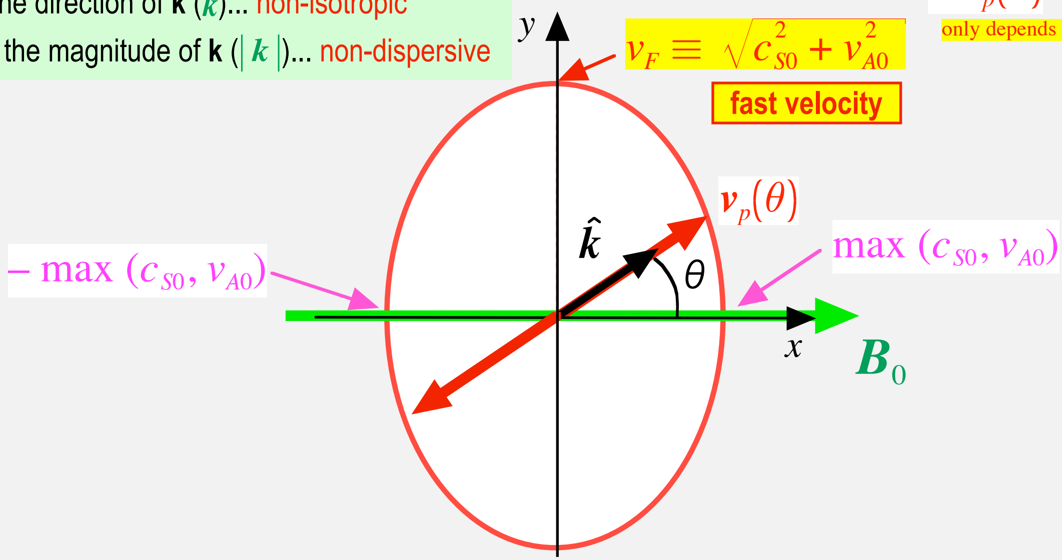
$$\boldsymbol{v}_p = \frac{\omega(k,\theta)}{k}\hat{\boldsymbol{k}} = \pm\sqrt{\frac{c_{S0}^2 + v_{A0}^2}{2} + \frac{\sqrt{c_{S0}^4 + v_{A0}^4 - 2\,c_{S0}^2 v_{A0}^2 \cos 2\theta}}{2}}\;\hat{\boldsymbol{k}}$$

$= \boldsymbol{v}_p(\theta)$

only depends on the direction of $\boldsymbol{k}$

|$\mathbf{v}_p$| is dependent on the direction of **k** ($\hat{k}$)... non-isotropic

|$\mathbf{v}_p$| is independent of the magnitude of **k** (| k |)... non-dispersive



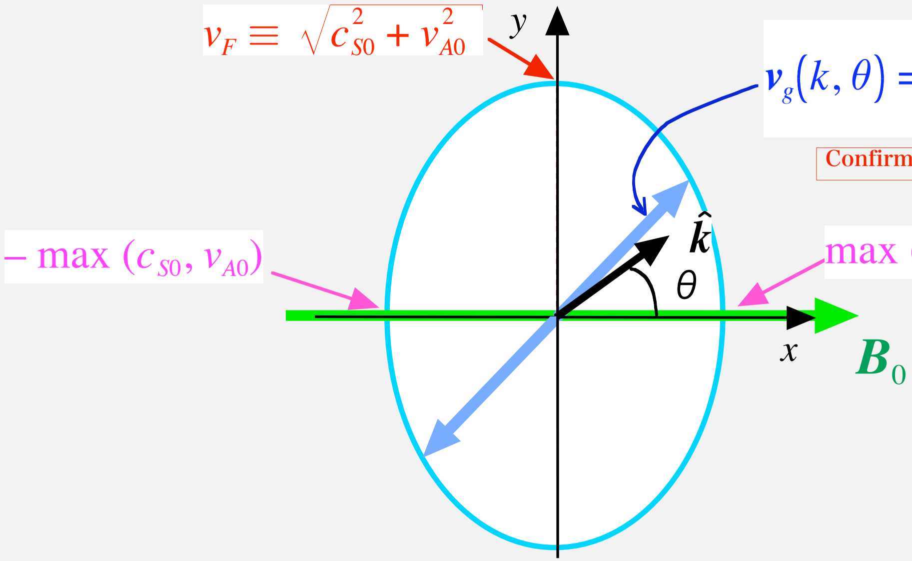
Group velocity

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}}\ , \quad \omega\left(k, \theta\right) = \pm\, k \sqrt{\frac{c_{S0}^2 + v_{A0}^2}{2} + \frac{\sqrt{c_{S0}^4 + v_{A0}^4 - 2\, c_{S0}^2\, v_{A0}^2 \cos 2\theta}}{2}}$$

$$k = \sqrt{k_x^2 + k_y^2}$$

$$\cos\left(2\theta\right) = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} = \frac{1 - \frac{k_y^2}{k_x^2}}{1 + \frac{k_y^2}{k_x^2}} = \frac{k_x^2 - k_y^2}{k_x^2 + k_y^2}$$

$$\omega\left(k_x, k_y\right)$$



$$\boldsymbol{v}_g\left(k, \theta\right) = \begin{pmatrix} \frac{\partial \omega}{\partial k_x} \\ \frac{\partial \omega}{\partial k_y} \end{pmatrix} = \boldsymbol{v}_g\left(\theta\right)$$

Confirm that $\boldsymbol{v}_g$ does not depend on $k = |\boldsymbol{k}|$.

Slow wave

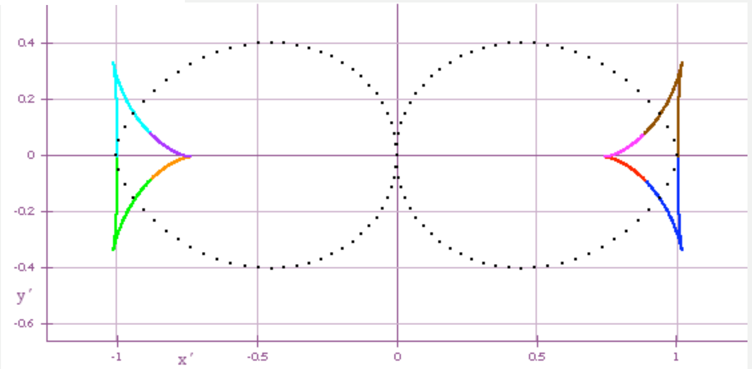
Vp (phase velocity)... dotted

Vg (group velocity)... colored

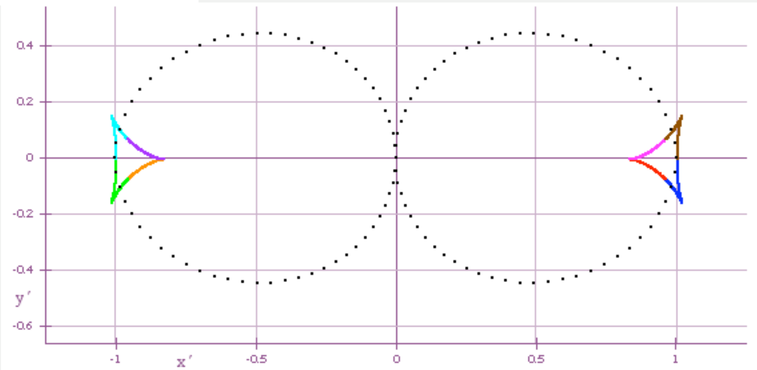
$0 \le \theta \le \frac{\pi}{4}$... *blue* , $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$... *red* , $\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$... *orange* , $\frac{3\pi}{4} \le \theta \le \pi$... *green*

$\pi \le \theta \le \frac{5\pi}{4}$... *cyan* , $\frac{5\pi}{4} \le \theta \le \frac{3\pi}{2}$... *purple* , $\frac{3\pi}{2} \le \theta \le \frac{7\pi}{4}$... *magenta* , $\frac{7\pi}{4} \le \theta \le 2\pi$... *brown*

$C_S = 1.0,\ V_A = 1.1$



$C_S = 1.0,\ V_A = 1.5$



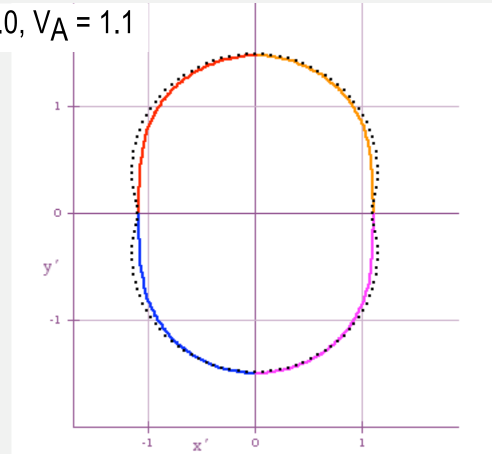
Fast wave

Vp (phase velocity)... dotted

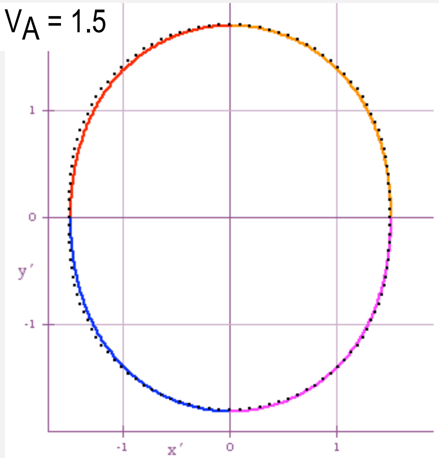
Vg (group velocity)... colored

$0 \le \theta \le \frac{\pi}{2}$... *orange* , $\frac{\pi}{2} \le \theta \le \pi$... *red* , $\pi \le \theta \le \frac{3\pi}{2}$... *blue* , $\frac{3\pi}{2} \le \theta \le 2\pi$... *magenta*

$C_S = 1.0,\ V_A = 1.1$

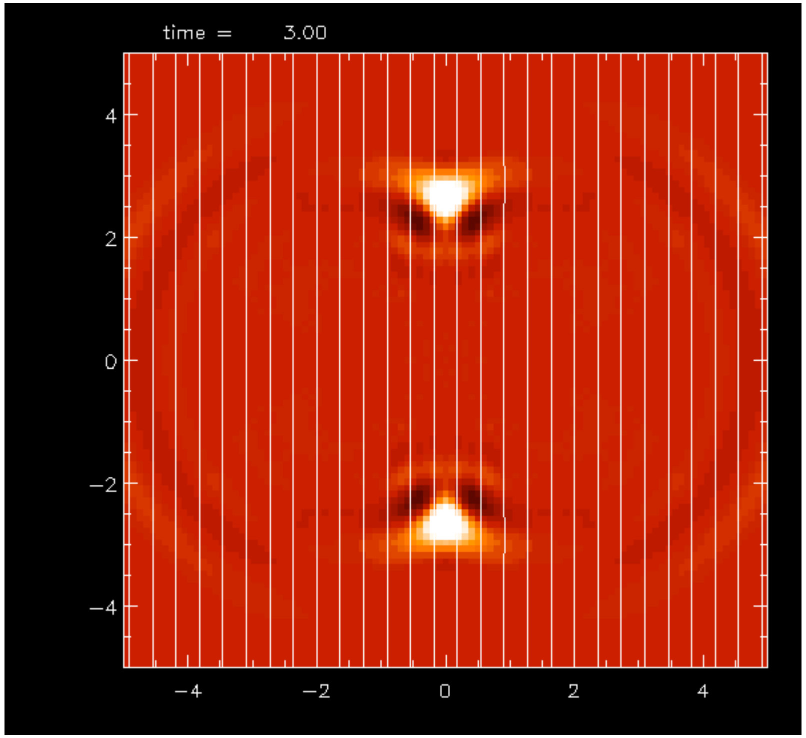


$C_S = 1.0,\ V_A = 1.5$

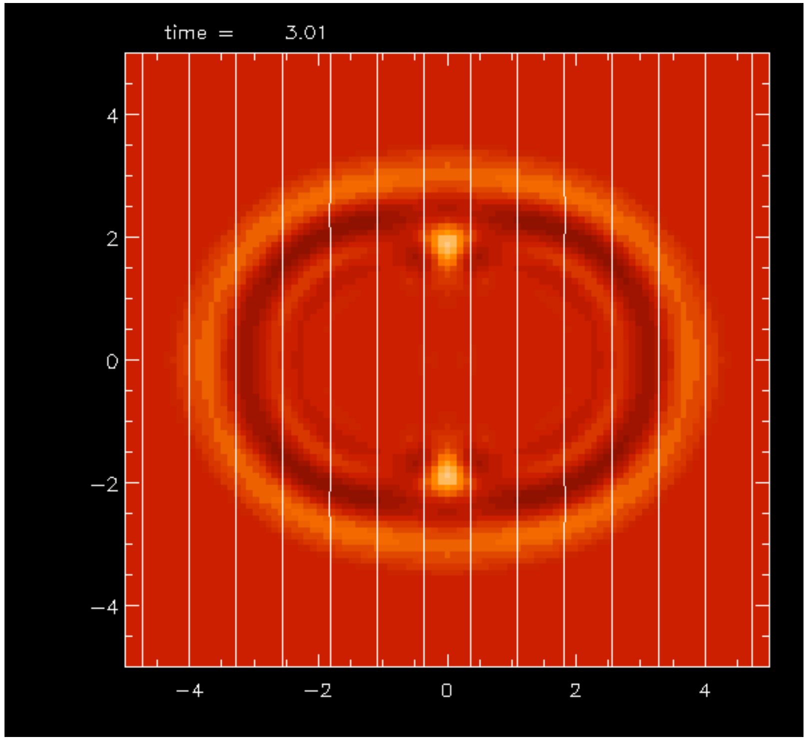


2-dimensional simulation of compressional MHD wave

Low plasma beta case (strong B)



High plasma beta case (weak B)

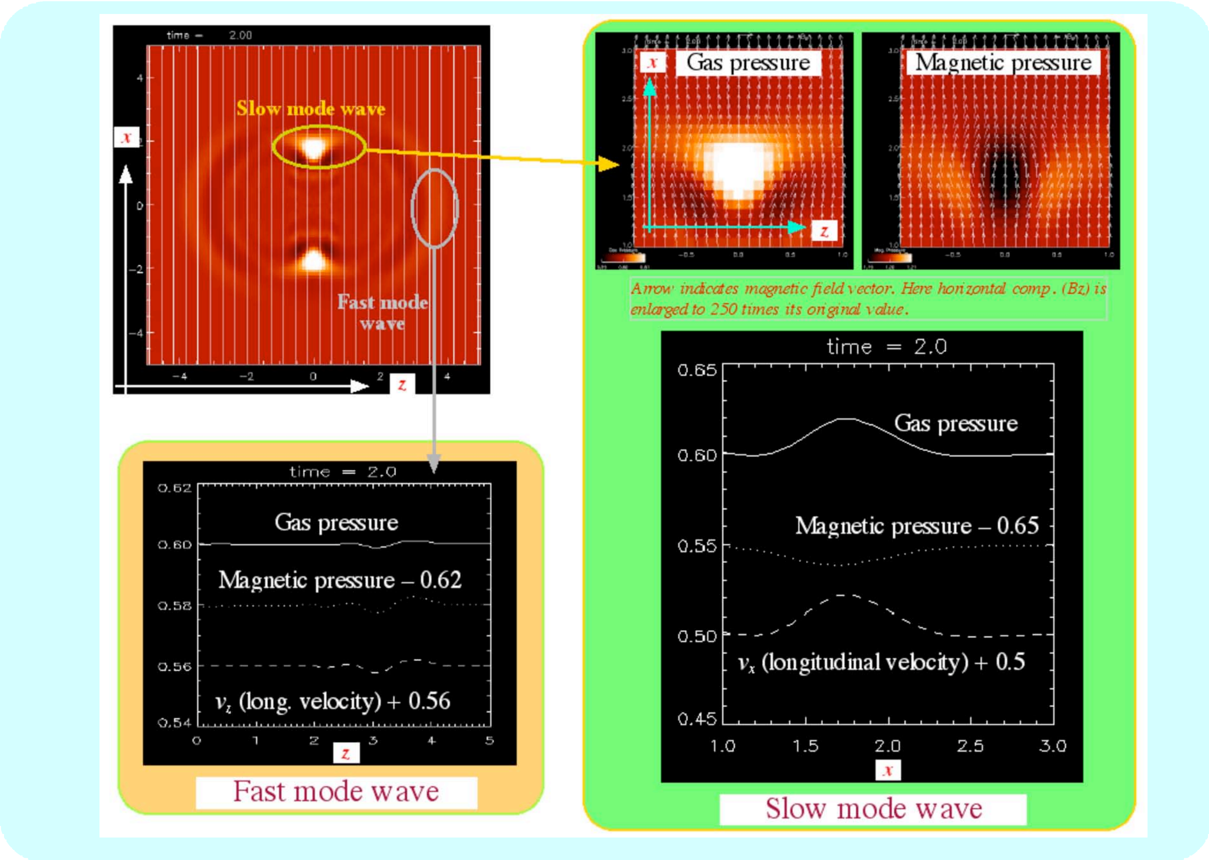


Oscillation continues at the origin for about 1.5-period.

Magnetic field plays a role in **transporting wave energy along it.**

For more details, see the website "http://163.180.179.74/~magara/page31/Topics/MHD_waves/mhdw.html".

Physical properties of compressional MHD wave:



gas pressure vs. magnetic pressure

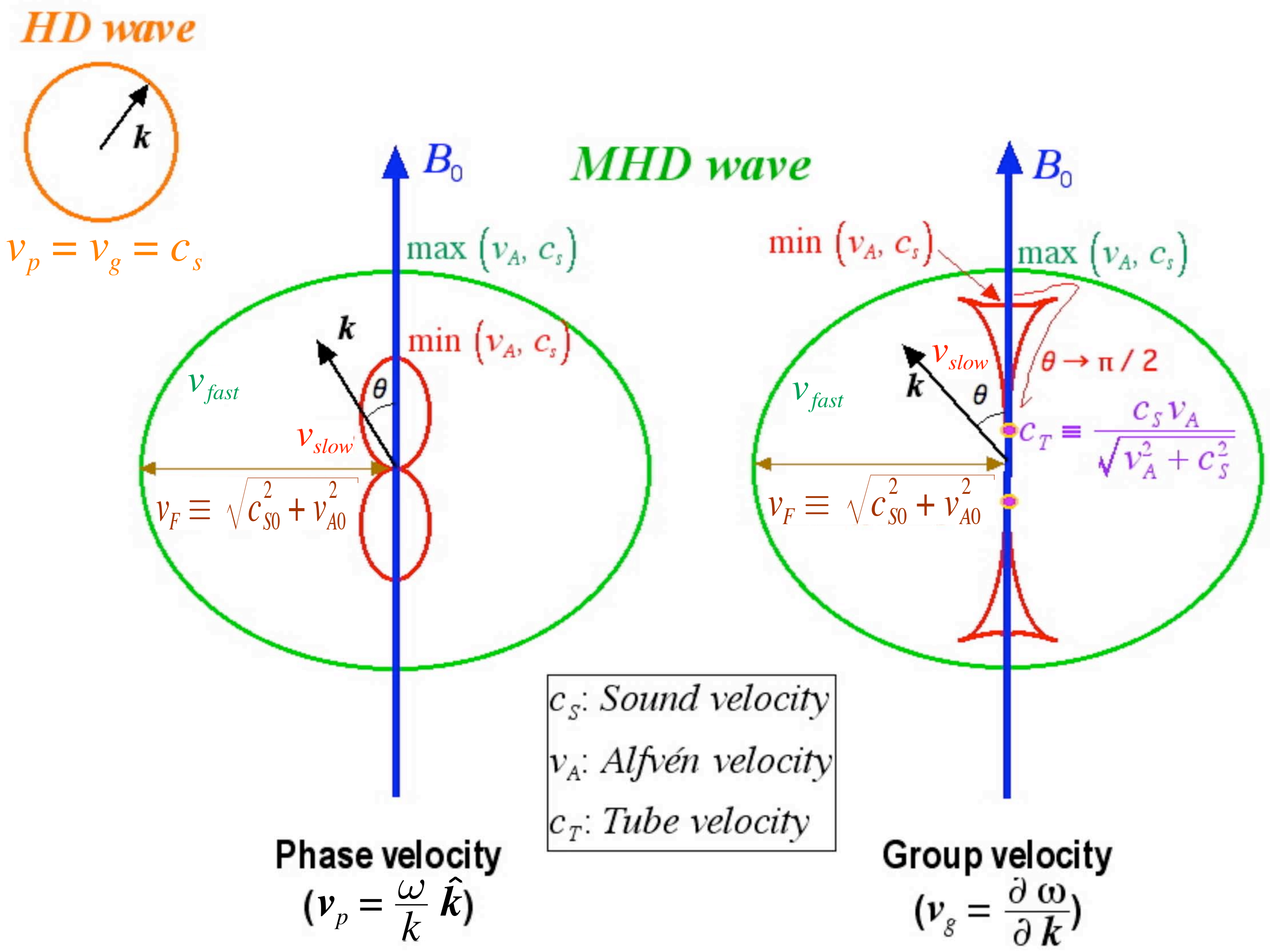
$$\left(\frac{\omega^2}{c_{s0}^2}-k^2\right)p^* = \frac{1}{\mu_0}\left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0\right)k^2$$

$\frac{\omega}{k} > c_{s0}$ (fast wave) => p^* *and* p_m^* *have positive correlation*

p^* *and* $\boldsymbol{v}^*$ *have positive correlation when* $\begin{cases} \boldsymbol{k} \perp \boldsymbol{B}_0 \text{ for fast–mode wave} \\ \boldsymbol{k} \,//\, \boldsymbol{B}_0 \text{ for slow–mode wave} \end{cases}$

$\frac{\omega}{k} < c_{s0}$ (slow wave) => p^* *and* p_m^* *have negative correlation*

Summary of compressional MHD wave



$$c_T < \min(c_{s0}, v_{A0}) < \max(c_{s0}, v_{A0}) < v_F$$