Dispersion relation: $\omega = \omega(\boldsymbol{k})$… relation between frequency and wavenumber vector

… represents physical properties of a wave

• Phase velocity: $\boldsymbol{v}_p = \dfrac{\omega}{k}\,\hat{\boldsymbol{k}}$

… propagation speed of **single-mode wave** ($k = k_0$ in 1D case)

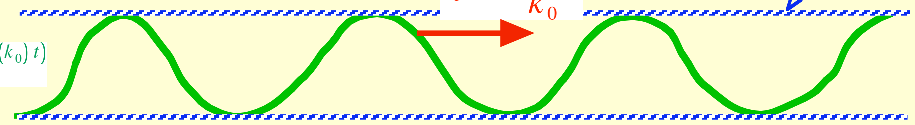
$\mathbf{v}_p$ always has **the same direction** as **k**.

1D case

$$\rho(r,t) = \rho^*(k_0)\, e^{i(k_0 r - \omega(k_0)\, t)}$$

$$v_p = \frac{\omega(k_0)}{k_0}$$

Wave amplitude
$\rho^*(k_0)$



• Group velocity: $\boldsymbol{v}_g = \dfrac{\partial\,\omega}{\partial\,\boldsymbol{k}}$

… propagation speed of the **envelope of multi-mode wave** ($k_0 - \Delta k \le k \le k_0 + \Delta k$ in 1D case)

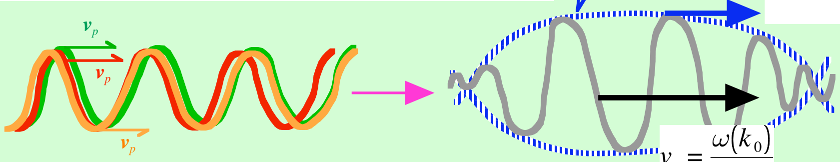
$\mathbf{v}_g$ does **not always have the same direction** as **k**, except for 1D case.

1D case

$$\rho(r,t) = \int \rho^*(k_0 + \Delta k')\, e^{i[(k_0+\Delta k')\, r - \omega(k_0+\Delta k')\, t]}\, d\Delta k' = G(r - v_g\, t)\, e^{i(k_0 r - \omega(k_0)\, t)}$$

Modulated wave amplitude
(it forms **wave packet**)
$G(r - v_g\, t)$

$$v_g = \left.\frac{\partial\,\omega}{\partial\,k}\right]_{k=k_0}$$



$$v_p = \frac{\omega(k_0)}{k_0}$$

Mathematical description of a multi-mode wave:

1D case

$$\rho(r,t) = \int \rho^*(k_0 + \Delta k')\, e^{i[(k_0+\Delta k')r - \omega(k_0+\Delta k')t]}\, d\Delta k'$$

$$(k_0 + \Delta k')\, r - \left[\omega(k_0) + \left.\frac{\partial \omega}{\partial k}\right|_{k=k_0} \Delta k'\right] t = k_0 r - \omega(k_0)\, t + \Delta k' \left[r - \left.\frac{\partial \omega}{\partial k}\right|_{k=k_0} t\right]$$

Taylor expansion (1st order)

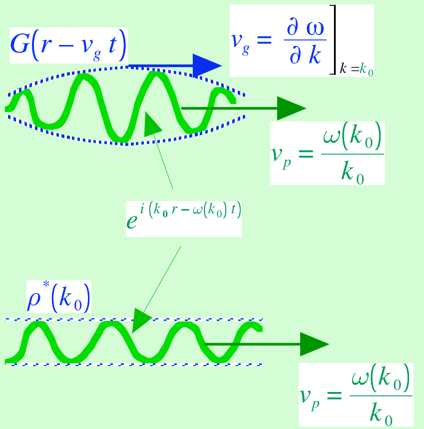
$$\rho^*(k_0 + \Delta k')\, e^{i \Delta k' \left[r - \left.\frac{\partial \omega}{\partial k}\right|_{k=k_0} t\right]}\, d\Delta k'\; e^{i(k_0 r - \omega(k_0) t)}$$

$$= G(r - v_g t)$$

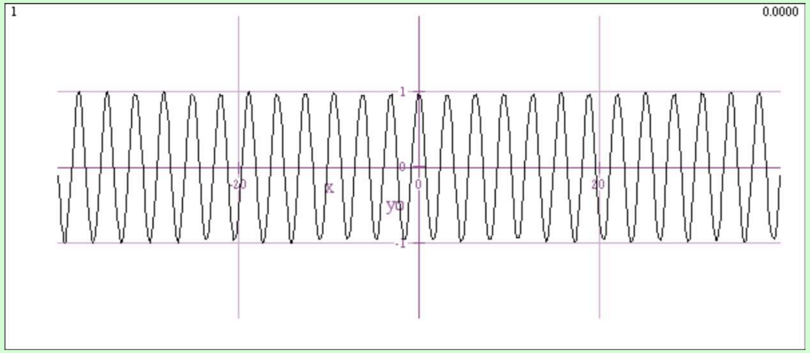
$$v_g = \left.\frac{\partial \omega}{\partial k}\right]_{k=k_0}$$

single-mode wave

$$\rho(r,t) = \rho^*(k_0)\, e^{i(k_0 r - \omega(k_0) t)}$$

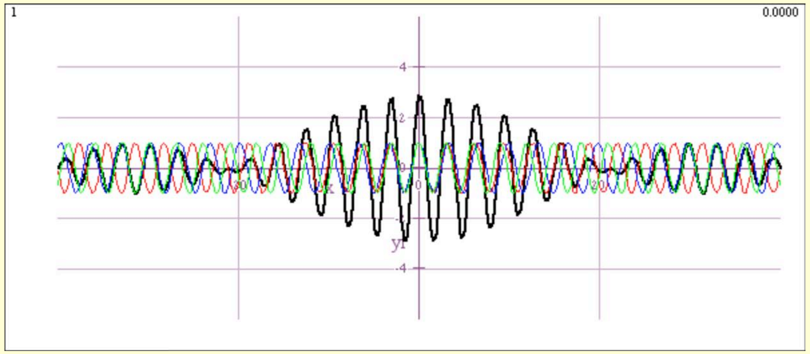


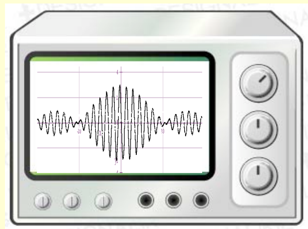
Propagation of single-mode wave



$\Delta k = 0 \;\Rightarrow\;$ *no energy, single–mode wave cannot be detected*

Propagation of multi-mode wave





$\Delta k \neq 0 \;\Rightarrow\;$ *non–zero energy, modulated wave (black pattern) can be detected*

Velocity diagram

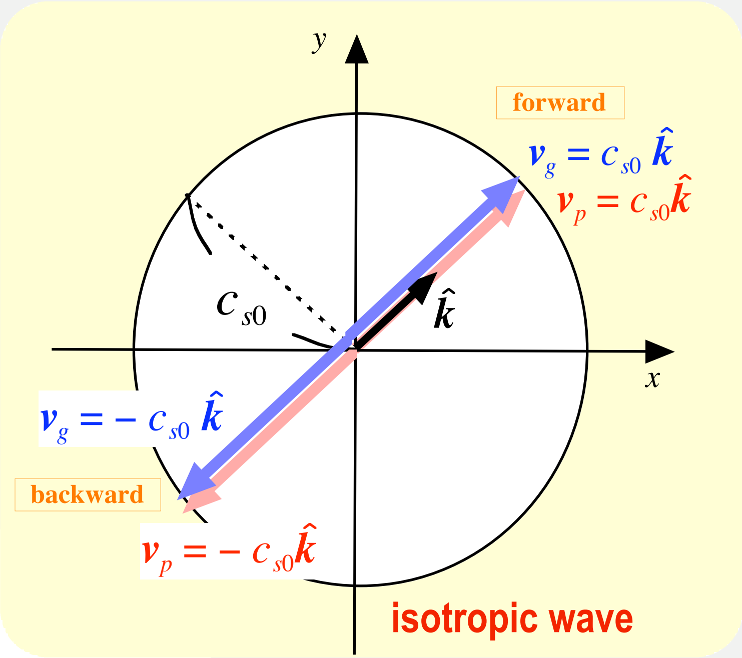
Dispersion relation of sound wave: $\omega(\boldsymbol{k}) = \pm k\, c_{s0}$

Phase velocity

$$\boldsymbol{v}_p = \frac{\omega}{k}\hat{\boldsymbol{k}} = \pm c_{s0}\hat{\boldsymbol{k}} \quad (+\text{... forward}, -\text{... backward})$$

$|\mathbf{v}_p|$ is independent of the direction of **k** ($\hat{k}$)... isotropic

$|\mathbf{v}_p|$ is independent of the magnitude of **k** ($|k|$)... non-dispersive



Group velocity

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}} = \begin{pmatrix} \frac{\partial \omega}{\partial k_x} \\ \frac{\partial \omega}{\partial k_y} \\ \frac{\partial \omega}{\partial k_z} \end{pmatrix}$$

$$= \pm c_{s0} \begin{pmatrix} \frac{\partial \sqrt{k_x^2 + k_y^2 + k_z^2}}{\partial k_x} \\ \frac{\partial \sqrt{k_x^2 + k_y^2 + k_z^2}}{\partial k_y} \\ \frac{\partial \sqrt{k_x^2 + k_y^2 + k_z^2}}{\partial k_z} \end{pmatrix} = \pm c_{s0} \begin{pmatrix} \frac{2k_x}{2\sqrt{k_x^2 + k_y^2 + k_z^2}} \\ \frac{2k_y}{2\sqrt{k_x^2 + k_y^2 + k_z^2}} \\ \frac{2k_z}{2\sqrt{k_x^2 + k_y^2 + k_z^2}} \end{pmatrix} = \pm c_{s0} \begin{pmatrix} \frac{k_x}{k} \\ \frac{k_y}{k} \\ \frac{k_z}{k} \end{pmatrix} = \pm c_{s0}\hat{\boldsymbol{k}}$$

Equal to the phase velocity

Physical properties of sound wave: (obtained from amplitude relations)

- $\omega \rho_0 \boldsymbol{v}^* = \boldsymbol{k} p^* \Rightarrow \boldsymbol{v}_1 // \boldsymbol{k}$... **longitudinal wave**

- $\boldsymbol{k} \bullet \boldsymbol{v}^* \neq 0 \Rightarrow \nabla \bullet \boldsymbol{v}_1 \neq 0$... **compressional wave**

- $p^* = c_{s0}^2 \rho^*$, $\omega \rho_0 \boldsymbol{v}^* = \boldsymbol{k} p^*$... **same phase** for $\rho_1, p_1, \boldsymbol{v}_1$

- $\rho^* = \frac{\rho_0}{\omega} \boldsymbol{k} \bullet \boldsymbol{v}^* \Rightarrow \rho_1 = \frac{\rho_0}{\omega} \boldsymbol{k} \bullet \boldsymbol{v}_1 = \frac{\rho_0}{\omega} (-i) \nabla \bullet \boldsymbol{v}_1$... **phase difference of π/2** for ρ_1 and $\nabla \bullet \boldsymbol{v}_1$

  $\nabla \bullet \boldsymbol{v}_1 = i\boldsymbol{k} \bullet \boldsymbol{v}_1$

Magnetic wave

$p_0 = 0$

Linearized ideal MHD equations

$$\begin{cases}\dfrac{\partial \rho_1(x,y,z,t)}{\partial t} = -\rho_0 \nabla \bullet \boldsymbol{v_1}(x,y,z,t) \\ \rho_0 \dfrac{\partial \boldsymbol{v_1}(x,y,z,t)}{\partial t} = -\nabla p_1(x,y,z,t) + \dfrac{1}{\mu_0}\left(\nabla \times \boldsymbol{B_1}(x,y,z,t)\right) \times \boldsymbol{B_0} \\ \dfrac{\partial p_1(x,y,z,t)}{\partial t} = \gamma \dfrac{p_0}{\rho_0} \dfrac{\partial \rho_1(x,y,z,t)}{\partial t} \\ \dfrac{\partial \boldsymbol{B}_1(x,y,z,t)}{\partial t} = \nabla \times \left(\boldsymbol{v_1}(x,y,z,t) \times \boldsymbol{B_0}\right)\end{cases}$$

$p_0 = 0,\ p_1 = 0$ →

$$\dfrac{\partial \rho_1(x,y,z,t)}{\partial t} = -\rho_0 \nabla \bullet \boldsymbol{v_1}(x,y,z,t)$$

$$\rho_0 \dfrac{\partial \boldsymbol{v_1}(x,y,z,t)}{\partial t} = \dfrac{1}{\mu_0}\left(\nabla \times \boldsymbol{B_1}(x,y,z,t)\right) \times \boldsymbol{B_0}$$

$$\dfrac{\partial \boldsymbol{B}_1(x,y,z,t)}{\partial t} = \nabla \times \left(\boldsymbol{v_1}(x,y,z,t) \times \boldsymbol{B_0}\right)$$

Dispersion relation of magnetic wave

$$\frac{\partial \rho_1(x,y,z,t)}{\partial t} = -\rho_0 \nabla \bullet \boldsymbol{v}_1(x,y,z,t)$$

$$\rho_0 \frac{\partial \boldsymbol{v}_1(x,y,z,t)}{\partial t} = \frac{1}{\mu_0}\left(\nabla \times \boldsymbol{B}_1(x,y,z,t)\right) \times \boldsymbol{B}_0$$

$$\frac{\partial \boldsymbol{B}_1(x,y,z,t)}{\partial t} = \nabla \times \left(\boldsymbol{v}_1(x,y,z,t) \times \boldsymbol{B}_0\right)$$

Partial differential equations

$$\rho_1(x,y,z,t) = \rho^* e^{i(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)}$$

$$\boldsymbol{v}_1(x,y,z,t) = \boldsymbol{v}^* e^{i(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)}$$

$$\boldsymbol{B}_1(x,y,z,t) = \boldsymbol{B}^* e^{i(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)}$$

$$-i\omega \rho^* e^{i(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)} = -\rho_0\, i\,\boldsymbol{k}\bullet\boldsymbol{v}^* e^{i(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)}$$

$$-i\omega \rho_0 \boldsymbol{v}^* e^{i(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)} = \frac{1}{\mu_0}\left[i\,\boldsymbol{k}\times\boldsymbol{B}^* e^{i(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)}\right]\times\boldsymbol{B}_0$$

$$-i\omega \boldsymbol{B}^* e^{i(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)} = i\,\boldsymbol{k}\times\left[\boldsymbol{v}^* e^{i(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)}\times\boldsymbol{B}_0\right]$$

e^i–form

$$\nabla \times ?_1 \Rightarrow i\,\boldsymbol{k}\times ?_1$$

$$\nabla \times \left(?_1 \times \boldsymbol{B}_0\right) \Rightarrow i\,\boldsymbol{k}\times\left(?_1\times\boldsymbol{B}_0\right)$$

when $\boldsymbol{B}_0$ is a constant vector

differential operation
=> algebraic operation

$$\omega\rho^* = \rho_0\,\boldsymbol{k}\bullet\boldsymbol{v}^*$$

$$\omega\rho_0\boldsymbol{v}^* = -\frac{1}{\mu_0}\left(\boldsymbol{k}\times\boldsymbol{B}^*\right)\times\boldsymbol{B}_0$$

$$\omega\boldsymbol{B}^* = -\boldsymbol{k}\times\left(\boldsymbol{v}^*\times\boldsymbol{B}_0\right)$$

Algebraic equations

$$\nabla \bullet \boldsymbol{B}_1(x,y,z,t) = 0 \quad\longrightarrow\quad i\,\boldsymbol{k}\bullet\boldsymbol{B}_1(x,y,z,t) = 0 \Rightarrow \boldsymbol{k}\bullet\boldsymbol{B}^* = 0$$

oscillation direction of magnetic field ⊥ propagation direction

... common property

$$\omega \rho^* = \rho_0 \, \boldsymbol{k} \bullet \boldsymbol{v}^*$$

$$\omega \rho_0 \boldsymbol{v}^* = -\frac{1}{\mu_0}\left[\boldsymbol{k} \times \boldsymbol{B}^*\right] \times \boldsymbol{B}_0 = -\frac{1}{\mu_0}\left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)\boldsymbol{B}^* - \left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0\right)\boldsymbol{k}\right]$$

$$\omega \boldsymbol{B}^* = -\boldsymbol{k} \times \left[\boldsymbol{v}^* \times \boldsymbol{B}_0\right] = -\left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)\boldsymbol{v}^* - \left(\boldsymbol{k} \bullet \boldsymbol{v}^*\right)\boldsymbol{B}_0\right]$$

triple vector product:
$(\boldsymbol{A} \times \boldsymbol{B}) \times \boldsymbol{C} = \boldsymbol{B}(\boldsymbol{A} \bullet \boldsymbol{C}) - \boldsymbol{A}(\boldsymbol{B} \bullet \boldsymbol{C})$

apply $\boldsymbol{k} \bullet$, $\boldsymbol{B}_0 \bullet$ *to both hands*

We have to consider two directions, $\boldsymbol{k}$ and $\boldsymbol{B}_0$.

$\bullet \boldsymbol{B}_0$

$$\left(\omega \rho_0 \boldsymbol{v}^*\right) \bullet \boldsymbol{B}_0 = \left(-\frac{1}{\mu_0}\left[\boldsymbol{k} \times \boldsymbol{B}^*\right] \times \boldsymbol{B}_0\right) \bullet \boldsymbol{B}_0$$

$= 0$

$$\boldsymbol{v}^* \bullet \boldsymbol{B}_0 = 0$$

oscillation direction of velocity ⊥ magnetic field

... common property

$\bullet \boldsymbol{k}$

$$\left(\omega \rho_0 \boldsymbol{v}^*\right) \bullet \boldsymbol{k} = \left(-\frac{1}{\mu_0}\left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)\boldsymbol{B}^* - \left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0\right)\boldsymbol{k}\right]\right) \bullet \boldsymbol{k}$$

$$\Rightarrow \omega \rho_0 \, \boldsymbol{k} \bullet \boldsymbol{v}^* = -\frac{1}{\mu_0}\left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)\left(\boldsymbol{k} \bullet \boldsymbol{B}^*\right) - \left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0\right)k^2\right]$$

$= 0$

$\bullet \boldsymbol{B}_0$

$$\omega \boldsymbol{B}^* \bullet \boldsymbol{B}_0 = -\left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)\boldsymbol{v}^* - \left(\boldsymbol{k} \bullet \boldsymbol{v}^*\right)\boldsymbol{B}_0\right] \bullet \boldsymbol{B}_0$$

$$\Rightarrow \omega \boldsymbol{B}^* \bullet \boldsymbol{B}_0 = -\left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)\boldsymbol{v}^* \bullet \boldsymbol{B}_0 - \left(\boldsymbol{k} \bullet \boldsymbol{v}^*\right)B_0^2\right]$$

$= 0$

eliminate $\boldsymbol{B}^* \bullet \boldsymbol{B}_0$

$$\left(\omega^2 - k^2 \frac{B_0^2}{\mu_0 \rho_0}\right)\left(\boldsymbol{k} \bullet \boldsymbol{v}^*\right) = 0$$

$= v_{A0}^2$ Alfvén speed

$\bullet \boldsymbol{k}$

$$0 = 0$$

$$\left(\omega^2 - k^2 \boxed{\frac{B_0^2}{\mu_0 \rho_0}}\right)(\boldsymbol{k} \bullet \boldsymbol{v}^*) = 0$$

$$= v_{A0}^2$$

1) $\boldsymbol{k} \bullet \boldsymbol{v}^* = 0$

2) $\boldsymbol{k} \bullet \boldsymbol{v}^* \neq 0$

1) $\boldsymbol{k} \bullet \boldsymbol{v}^* = 0 \Rightarrow \nabla \bullet \boldsymbol{v}_1 = 0$ ... *incompressional wave* => **Shear Alfvén wave**

$$\omega \rho^* = -\rho_0 \boldsymbol{k} \bullet \boldsymbol{v}^* \rightarrow \rho^* = 0$$

$$\omega \rho_0 \boldsymbol{v}^* = -\frac{1}{\mu_0}\left[\boldsymbol{k} \times \boldsymbol{B}^*\right] \times \boldsymbol{B}_0 = -\frac{1}{\mu_0}\left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)\boldsymbol{B}^* - \left(\boldsymbol{B}^* \bullet \boldsymbol{B}_0\right)\boldsymbol{k}\right]$$

$$\omega \boldsymbol{B}^* = -\boldsymbol{k} \times \left[\boldsymbol{v}^* \times \boldsymbol{B}_0\right] = -\left[\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)\boldsymbol{v}^* - \underbrace{\left(\boldsymbol{k} \bullet \boldsymbol{v}^*\right)}_{=0}\boldsymbol{B}_0\right]$$

eliminate $\boldsymbol{B}^*$

$$\omega^2 \rho_0 \boldsymbol{v}^* = -\frac{1}{\mu_0}\left[-\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)^2 \boldsymbol{v}^* + \left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)\underbrace{\left(\boldsymbol{v}^* \bullet \boldsymbol{B}_0\right)}_{=0}\boldsymbol{k}\right]$$

... common property

$$\omega^2 \rho_0 \boldsymbol{v}^* = -\frac{1}{\mu_0}\left[-\left(\boldsymbol{k} \bullet \boldsymbol{B}_0\right)^2 \boldsymbol{v}^*\right]$$

$$\omega = \pm \frac{\boldsymbol{k} \bullet \boldsymbol{B}_0}{\sqrt{\mu_0 \rho_0}} \rightarrow \omega = \pm k\, v_{A0} \cos\theta$$

dispersion relation

Velocity diagram

Dispersion relation of shear Alfvén wave: $\omega(\boldsymbol{k}) = \pm \dfrac{\boldsymbol{k} \bullet \boldsymbol{B}_0}{\sqrt{\mu_0 \rho_0}} = \pm k\, v_{A0} \cos\theta$

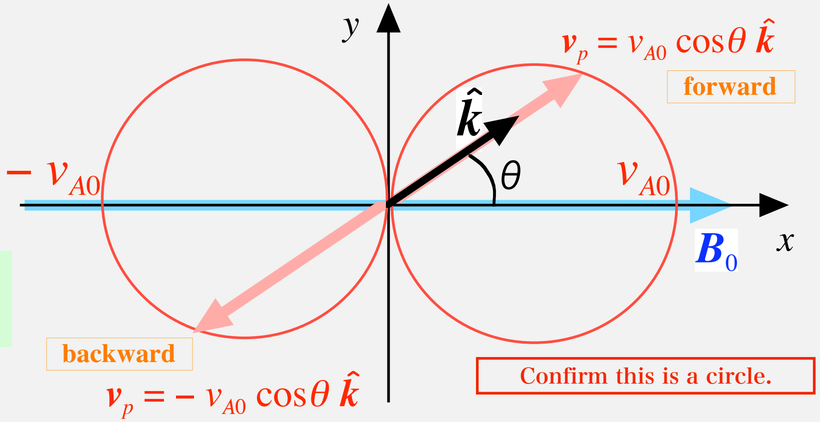
Phase velocity

$$\boldsymbol{v}_p = \frac{\omega}{k}\hat{\boldsymbol{k}} = \pm v_{A0} \cos\theta\, \hat{\boldsymbol{k}}$$

(+… forward, −… backward)

$|\mathbf{v}_p|$ is dependent on the direction of **k** ($\hat{k}$)… non-isotropic

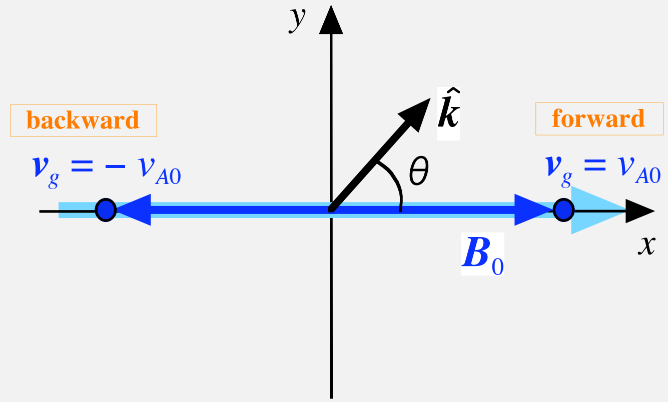
$|\mathbf{v}_p|$ is independent of the magnitude of **k** ($|\,k\,|$)… non-dispersive



Group velocity

$$\boldsymbol{v}_g = \frac{\partial \omega}{\partial \boldsymbol{k}} = \pm \begin{pmatrix} \frac{\partial}{\partial k_x} \\ \frac{\partial}{\partial k_y} \\ \frac{\partial}{\partial k_z} \end{pmatrix} \frac{\boldsymbol{k} \bullet \boldsymbol{B}_0}{\sqrt{\mu_0 \rho_0}}$$

$$= \pm \begin{pmatrix} \frac{\partial}{\partial k_x} \\ \frac{\partial}{\partial k_y} \\ \frac{\partial}{\partial k_z} \end{pmatrix} \frac{k_x B_0 + k_y 0 + k_z 0}{\sqrt{\mu_0 \rho_0}} = \pm \begin{pmatrix} \frac{B_0}{\sqrt{\mu_0 \rho_0}} \\ 0 \\ 0 \end{pmatrix} = \pm \begin{pmatrix} v_{A0} \\ 0 \\ 0 \end{pmatrix}$$



Energy (wave packet) is always transported along magnetic field $\boldsymbol{B}_0$.