3. $\frac{\partial}{\partial t}\left(\frac{p}{\gamma - 1}\right) + \nabla \bullet \left(\frac{p}{\gamma - 1}\,\boldsymbol{v}\right) = -\,p\,\nabla \bullet \boldsymbol{v}$

$$\frac{\partial}{\partial t}\left(\frac{p}{\gamma - 1}\right) + \nabla \bullet \left(\frac{p}{\gamma - 1}\,\boldsymbol{v}\right) = -\,p\,\nabla \bullet \boldsymbol{v}$$

$$\frac{p}{\gamma - 1}\,\nabla \bullet \boldsymbol{v} + \boldsymbol{v} \bullet \nabla \left(\frac{p}{\gamma - 1}\right)$$

$$\nabla \bullet \boldsymbol{v} = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + \boldsymbol{v} \bullet \nabla\,\rho\right)$$

mass conservation

$$\frac{\partial p}{\partial t} + \boldsymbol{v} \bullet \nabla\,p - \frac{\gamma\,p}{\rho}\left[\frac{\partial \rho}{\partial t} + \boldsymbol{v} \bullet \nabla\,\rho\right] = 0$$

$\rho_0 + \rho_1\,(x, y, z, t)$

$\mathbf{0} + \boldsymbol{v}_1\,(x, y, z, t)$

$p_0 + p_1\,(x, y, z, t)$

$$\frac{\partial p}{\partial t} + \boldsymbol{v} \bullet \nabla p - \frac{\gamma p}{\rho}\left[\frac{\partial \rho}{\partial t} + \boldsymbol{v} \bullet \nabla \rho\right] = 0$$

$$\frac{\partial \rho_1(x,y,z,t)}{\partial t} + \boldsymbol{v}_1(x,y,z,t) \bullet \nabla \rho_0$$

$= 0$

(ρ_0 is constant)

$$\frac{\partial\left(p_0 + p_1(x,y,z,t)\right)}{\partial t} + \boldsymbol{v}_1(x,y,z,t) \bullet \nabla\left(p_0 + p_1(x,y,z,t)\right)$$

$$\frac{\partial p_1(x,y,z,t)}{\partial t}$$

2nd-order

$\square_1 \bigcirc_1 \fallingdotseq 0$

$$\boldsymbol{v}_1(x,y,z,t) \bullet \nabla p_0 = 0 \quad (p_0 \textit{ is constant})$$

$$\gamma \frac{p_0 + p_1(x,y,z,t)}{\rho_0 + \rho_1(x,y,z,t)} = \gamma\left(p_0 + p_1(x,y,z,t)\right)\left(\rho_0 + \rho_1(x,y,z,t)\right)^{-1}$$

$$= \gamma \frac{p_0}{\rho_0}\left(1 + \frac{p_1(x,y,z,t)}{p_0}\right)\left(1 + \frac{\rho_1(x,y,z,t)}{\rho_0}\right)^{-1}$$

$$(1+x)^{-1} \approx 1 - x \quad when\ |x| \ll 1$$

$$\approx \gamma \frac{p_0}{\rho_0}\left(1 + \frac{p_1(x,y,z,t)}{p_0}\right)\left(1 - \frac{\rho_1(x,y,z,t)}{\rho_0} + \ldots\right)$$

$$\left(\frac{p_1(x,y,z,t)}{p_0} \times \frac{\rho_1(x,y,z,t)}{\rho_0} \approx 0\right)$$

$$\approx \gamma \frac{p_0}{\rho_0}\left(1 + \frac{p_1(x,y,z,t)}{p_0} - \frac{\rho_1(x,y,z,t)}{\rho_0}\right)$$

$$\frac{\partial p}{\partial t} + \boldsymbol{v} \bullet \nabla p - \frac{\gamma p}{\rho}\left[\frac{\partial \rho}{\partial t} + \boldsymbol{v} \bullet \nabla \rho\right] = 0$$

$$\Rightarrow \quad \frac{\partial p_1(x,y,z,t)}{\partial t} = \gamma \frac{p_0}{\rho_0}\left(1 + \frac{p_1(x,y,z,t)}{p_0} - \frac{\rho_1(x,y,z,t)}{\rho_0}\right)\frac{\partial \rho_1(x,y,z,t)}{\partial t}$$

Only 1st-order terms remain

$$\frac{\partial p_1(x,y,z,t)}{\partial t} = \gamma \frac{p_0}{\rho_0} \frac{\partial \rho_1(x,y,z,t)}{\partial t}$$

Linearized internal energy equation

4. $\dfrac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$

$$\boldsymbol{0} + \boldsymbol{v}_1(x, y, z, t)$$
$$\boldsymbol{B}_0 + \boldsymbol{B}_1(x, y, z, t)$$

$$\longrightarrow \quad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$$

$$\frac{\partial\left(\boldsymbol{B}_0 + \boldsymbol{B}_1(x, y, z, t)\right)}{\partial t} = \nabla \times \left[\boldsymbol{v}_1(x, y, z, t) \times \left(\boldsymbol{B}_0 + \boldsymbol{B}_1(x, y, z, t)\right)\right]$$

2nd-order: $\boldsymbol{v}_1 \times \boldsymbol{B}_1 \approx \boldsymbol{0}$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} \qquad \nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0)$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla \times \;\; \boldsymbol{v}_1 \times \boldsymbol{B}_0$$

Linearized induction equation

Ideal MHD equations

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \bullet (\rho\, \boldsymbol{v}) = 0 \\ \rho \left(\dfrac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \bullet \nabla\, \boldsymbol{v} \right) = -\nabla p + \dfrac{1}{\mu_0} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} \\ \rho \dfrac{d}{dt} \left(\dfrac{1}{\gamma - 1} \dfrac{p}{\rho} \right) + p\, \nabla \bullet \boldsymbol{v} = 0 \\ \dfrac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \end{cases}$$

$$\nabla \bullet \boldsymbol{B} = 0$$

$$\begin{aligned} &\rho_0 + \rho_1 (x, y, z, t) \\ &\mathbf{0} + \boldsymbol{v}_1 (x, y, z, t) \\ &p_0 + p_1 (x, y, z, t) \\ &\boldsymbol{B}_0 + \boldsymbol{B}_1 (x, y, z, t) \end{aligned}$$

Linearized ideal MHD equations

$$\begin{cases} \dfrac{\partial \rho_1 (x, y, z, t)}{\partial t} = -\rho_0 \nabla \bullet \boldsymbol{v}_1 (x, y, z, t) \\ \rho_0 \dfrac{\partial \boldsymbol{v}_1 (x, y, z, t)}{\partial t} = -\nabla p_1 (x, y, z, t) + \dfrac{1}{\mu_0} \left(\nabla \times \boldsymbol{B}_1 (x, y, z, t)\right) \times \boldsymbol{B}_0 \\ \dfrac{\partial p_1 (x, y, z, t)}{\partial t} = \gamma \dfrac{p_0}{\rho_0} \dfrac{\partial \rho_1 (x, y, z, t)}{\partial t} \\ \dfrac{\partial \boldsymbol{B}_1 (x, y, z, t)}{\partial t} = \nabla \times \left(\boldsymbol{v}_1 (x, y, z, t) \times \boldsymbol{B}_0\right) \end{cases}$$

$$\nabla \bullet \boldsymbol{B}_1 (x, y, z, t) = 0$$

Mathematical description of a linear wave (e^i–form: $u^* f(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t+\phi_0) \Rightarrow |u^*| e^{i\phi_0} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$)

linear wave solution = combination of $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ (Fourier decomposition)

$$\rho_1(x,y,z,t) = \rho^* e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \qquad \rho^* = |\rho^*| e^{i\alpha}$$

$$\boldsymbol{v}_1(x,y,z,t) = \boldsymbol{v}^* e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \qquad v_j^* = |v_j^*| e^{i\beta_j},\ j=x,y,z$$

$$p_1(x,y,z,t) = p^* e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \qquad p^* = |p^*| e^{i\delta}$$

$$\boldsymbol{B}_1(x,y,z,t) = \boldsymbol{B}^* e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} \qquad B_j^* = |B_j^*| e^{i\varepsilon_j},\ j=x,y,z$$

$\alpha, \beta_j, \delta, \varepsilon_j$... initial phase of each quantity

e.g. $\rho^* e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = |\rho^*| e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t+\alpha)}$

We always take the **real part** of this form as a linear wave solution:

$$\mathrm{Re}\left[e^{i\phi_0} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right] = \cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t+\phi_0)$$

As long as **linear term** is considered, we can **always retrieve a correct result** by **taking the real part**:

e.g., $\dfrac{d}{dx}\cos x = \boxed{-\sin x}$

$$\frac{d}{dx} e^{ix} = i\,e^{ix} = i(\cos x + i\sin x) = \boxed{-\sin x} + i\cos x$$

※ e^i–*form* **cannot be used for nonlinear term** (confirm this by considering $(\cos x)^2 \Leftrightarrow (e^{ix})^2$).

Sound wave

$$\boldsymbol{B}_0 = \boldsymbol{0}$$

Linearized ideal MHD equations

$$
\begin{cases}
\dfrac{\partial \rho_1(x,y,z,t)}{\partial t} = -\rho_0 \nabla \bullet \boldsymbol{v_1}(x,y,z,t) \\
\rho_0 \dfrac{\partial \boldsymbol{v_1}(x,y,z,t)}{\partial t} = -\nabla p_1(x,y,z,t) + \dfrac{1}{\mu_0}\left(\nabla \times \boldsymbol{B_1}(x,y,z,t)\right) \times \boldsymbol{B_0} \\
\dfrac{\partial p_1(x,y,z,t)}{\partial t} = \gamma \dfrac{p_0}{\rho_0} \dfrac{\partial \rho_1(x,y,z,t)}{\partial t} \\
\dfrac{\partial \boldsymbol{B}_1(x,y,z,t)}{\partial t} = \nabla \times \left(\boldsymbol{v_1}(x,y,z,t) \times \boldsymbol{B_0}\right)
\end{cases}
$$

$\boldsymbol{B}_0 = \mathbf{0}, \boldsymbol{B}_1 = \mathbf{0}$ →

$$
\frac{\partial \rho_1(x,y,z,t)}{\partial t} = -\rho_0 \nabla \bullet \boldsymbol{v_1}(x,y,z,t)
$$

$$
\rho_0 \frac{\partial \boldsymbol{v_1}(x,y,z,t)}{\partial t} = -\nabla p_1(x,y,z,t)
$$

$$
\frac{\partial p_1(x,y,z,t)}{\partial t} = \gamma \frac{p_0}{\rho_0} \frac{\partial \rho_1(x,y,z,t)}{\partial t}
$$

Dispersion relation of sound wave

$$\frac{\partial \rho_1(x,y,z,t)}{\partial t} = -\rho_0 \nabla \bullet \boldsymbol{v}_1(x,y,z,t)$$

$$\rho_0 \frac{\partial \boldsymbol{v}_1(x,y,z,t)}{\partial t} = -\nabla p_1(x,y,z,t)$$

$$\frac{\partial p_1(x,y,z,t)}{\partial t} = \gamma \frac{p_0}{\rho_0} \frac{\partial \rho_1(x,y,z,t)}{\partial t}$$

Partial differential equations

$$\rho_1(x,y,z,t) = \rho^* e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$$

$$\boldsymbol{v}_1(x,y,z,t) = \boldsymbol{v}^* e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$$

$$p_1(x,y,z,t) = p^* e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}$$

e^i–form

$$\frac{\partial}{\partial t} ?_1 \Rightarrow -i\omega ?_1$$

$$\nabla \bullet ?_1 \Rightarrow i\boldsymbol{k} \bullet ?_1$$

$$\nabla ?_1 \Rightarrow i\boldsymbol{k} ?_1$$

differential operation
=> algebraic operation

$$-i\omega\rho^* = -\rho_0 i \boldsymbol{k} \bullet \boldsymbol{v}^*$$

$$-i\omega\rho_0 \boldsymbol{v}^* = -i\boldsymbol{k} p^*$$

$$-i\omega p^* = \gamma \frac{p_0}{\rho_0}\left(-i\omega\rho^*\right)$$

Algebraic equations

$\omega \rho_0 \boldsymbol{v}^* = \boldsymbol{k} p^* \Rightarrow \boldsymbol{v}^* // \boldsymbol{k}$ ... *longitudinal wave*

apply $\boldsymbol{k} \bullet$ to both hands

Gas pressure gradient force is isotropic when p_0 is uniform, so we only consider the direction of $\boldsymbol{k}$.

$$-\omega\rho^* = -\rho_0 \boldsymbol{k} \bullet \boldsymbol{v}^*$$

$$\omega\rho_0 \boldsymbol{k} \bullet \boldsymbol{v}^* = k^2 p^*$$

$$p^* = \gamma \frac{p_0}{\rho_0} \rho^*$$

$$\begin{bmatrix} -\omega & \rho_0 & 0 \\ 0 & \omega\rho_0 & -k^2 \\ \gamma \frac{p_0}{\rho_0} & 0 & -1 \end{bmatrix} \begin{bmatrix} \rho^* \\ \boldsymbol{k}\bullet\boldsymbol{v}^* \\ p^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\omega & \rho_0 & 0 \\ 0 & \omega\rho_0 & -k^2 \\ \gamma\dfrac{p_0}{\rho_0} & 0 & -1 \end{bmatrix}\begin{bmatrix} \rho^* \\ \boldsymbol{k}\bullet\boldsymbol{v}^* \\ p^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boldsymbol{k}\bullet\boldsymbol{v}^* \neq 0 \Rightarrow \nabla\bullet\boldsymbol{v}_1 \neq 0$ ... *compressional wave*

if $\boldsymbol{k}\bullet\boldsymbol{v}^* = 0$, then

$$\rho^* = \frac{\rho_0}{\omega}\boldsymbol{k}\bullet\boldsymbol{v}^* = 0$$

$$p^* = \gamma\frac{p_0}{\rho_0}\rho^* = 0$$

$$\begin{bmatrix} \rho^* \\ \boldsymbol{k}\bullet\boldsymbol{v}^* \\ p^* \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \textit{requires} \quad \det\begin{bmatrix} -\omega & \rho_0 & 0 \\ 0 & \omega\rho_0 & -k^2 \\ \gamma\dfrac{p_0}{\rho_0} & 0 & -1 \end{bmatrix} = 0$$

$$\det\begin{bmatrix} -\omega & \rho_0 & 0 \\ 0 & \omega\rho_0 & -k^2 \\ \gamma\dfrac{p_0}{\rho_0} & 0 & -1 \end{bmatrix} = -\omega\,\omega\,\rho_0(-1) + (\rho_0)\left(-k^2\right)\left(\gamma\frac{p_0}{\rho_0}\right) - 0 = \omega^2\rho_0 - \gamma\frac{p_0}{\rho_0}\rho_0 k^2 = 0$$

$$\therefore\ \omega^2 - k^2 c_{s0}^2 = 0, \quad c_{s0} = \sqrt{\gamma\frac{p_0}{\rho_0}} \ \textit{(adiabatic sound speed)}$$