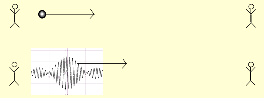
What is a wave?

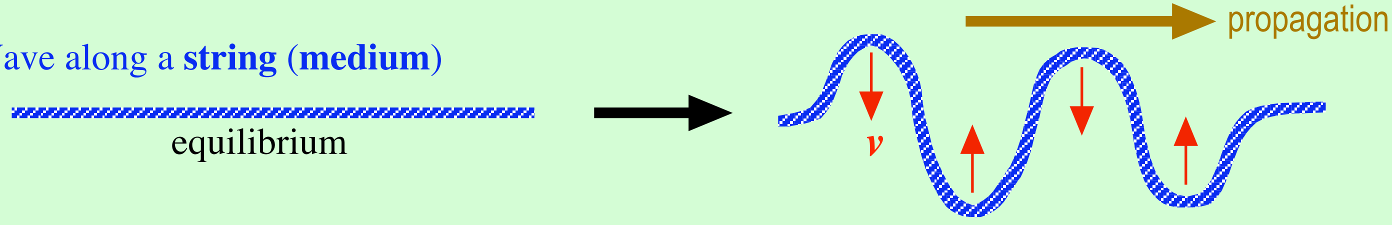
It is a time-dependent phenomenon in which a **physical state** is transported from one location to other locations through a medium (what is transported is **physical state**, not physical object).



Example:

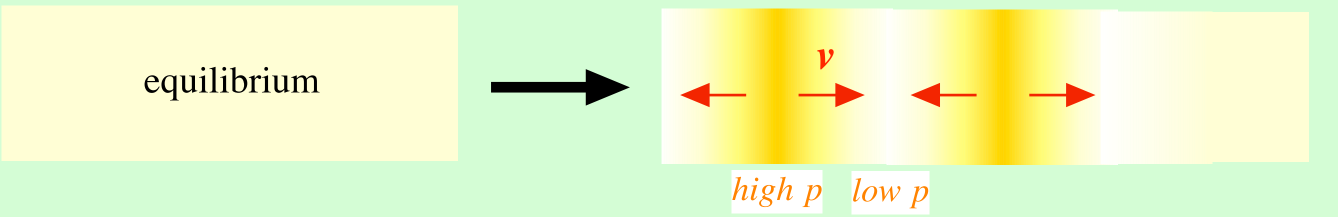
Transverse wave... **oscillation direction ⊥ propagation direction**

Wave along a **string** (**medium**)



Longitudinal wave... **oscillation direction // propagation direction**

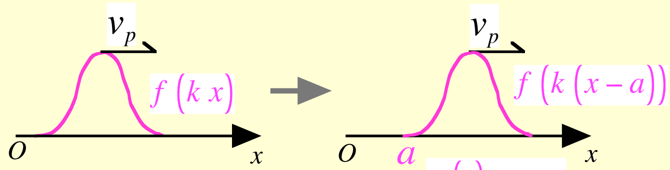
Sound wave in the **air** (**medium**)



Mathematical description of a wave

1-dimensional case

$$f\left(k\left(x-a\left(t\right)\right)\right)=f\left(k\left(x-\boxed{v_p}\,t\right)\right)=f\left(k\,x-\omega\,t\right)$$



$f\left(?\right)$ => *profile of a wave*

$a\left(t\right)=v_p\,t$

v_p: *phase speed*

k: *wavenumber* $\left(\lambda=\frac{2\pi}{k}\text{: wavelength}\right)$

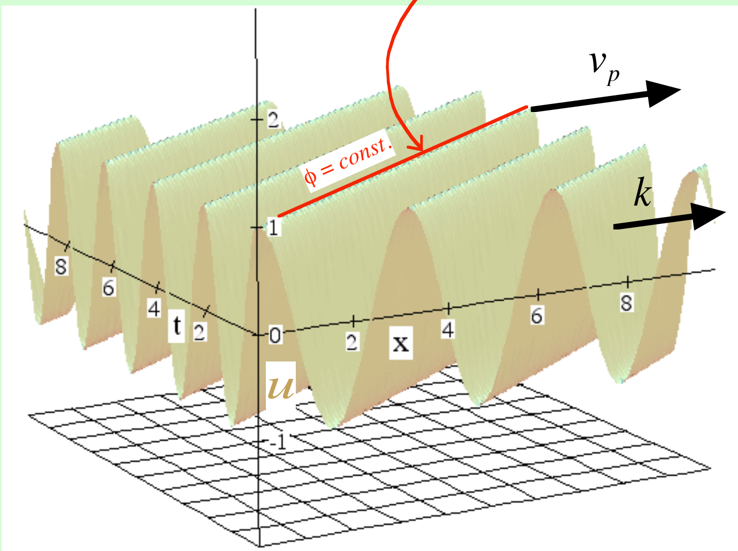
$\omega=k\,v_p$: *angular frequency* $\left(T=\frac{2\pi}{\omega}\text{: period}\right)$

$\phi\equiv k\,x-\omega\,t$: *phase* $\Rightarrow$ physical state

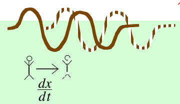
Example: $u\left(x,t\right)=f\left(k\,x-\omega\,t\right)=\cos\left(2\,x-3\,t\right)$

Trajectory of the same phase (= the same physical state) in (x, t)-space:

$$\phi\equiv k\,x-\omega\,t=const.$$



$$\frac{d\phi}{dt}=0\ \text{(Lagrangian derivative)}\ \Rightarrow\ k\frac{dx}{dt}-\omega=0$$



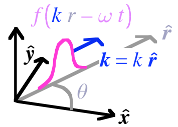
$$v_p=\frac{dx}{dt}=\frac{\omega}{k}\text{: phase speed}$$

Propagation speed of the same phase (the same physical state)

$$\cos\left(2\,x-3\,t\right)=\cos\left(2\left[x-\boxed{\frac{3}{2}}\,t\right]\right)$$

The profile of "$\cos\left(2x\right)$" moves toward positive x-direction at a speed of 3/2.

3-dimensional case



$$
\begin{aligned}
k\,r &= k\,\hat{\boldsymbol{r}}\cdot\boldsymbol{r} \\
&= k\left(\cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{y}}\right)\cdot\left(x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}}\right) \\
&= k_x\,x + k_y\,y = \boldsymbol{k}\bullet\boldsymbol{r}
\end{aligned}
$$

$$u\,(x,y,z,t) = u^{*}\,f\left(\boldsymbol{k}\bullet\boldsymbol{r} - \omega\,t + \phi_0\right)$$

u^{*}: *amplitude* $\quad \boldsymbol{r} = (x, y, z) \quad \boldsymbol{k} = (k_x, k_y, k_z)$: *wavenumber vector* => **propagation direction**

ω: *angular frequency* $\quad \boldsymbol{v}_p = \dfrac{\omega}{k}\,\hat{\boldsymbol{k}}$: *phase velocity* $\quad \phi \equiv \boldsymbol{k}\bullet\boldsymbol{r} - \omega\,t + \phi_0$: *phase*

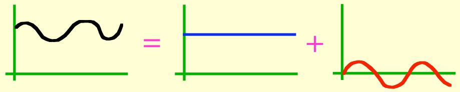
$$\boldsymbol{k}\bullet\boldsymbol{r} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}\bullet\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_x\,x + k_y\,y + k_z\,z$$

$$0 = \frac{d\phi}{dt} = \frac{d}{dt}\left(\boldsymbol{k}\bullet\boldsymbol{r} - \omega\,t + \phi_0\right) \;\Rightarrow\; \boldsymbol{k}\bullet\frac{d\boldsymbol{r}}{dt} - \omega = 0 \;\Rightarrow\; \boldsymbol{k}\bullet\boldsymbol{v}_p - \omega = 0 \;\Rightarrow\; k\,v_p = \omega \;\; (\text{because of } \boldsymbol{k}\;//\;\boldsymbol{v}_p)$$

$$\boldsymbol{v}_p = \frac{\omega}{k}\,\hat{k} = \frac{\omega}{\sqrt{k_x^2 + k_y^2 + k_z^2}}\begin{pmatrix} \dfrac{k_x}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \\ \dfrac{k_y}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \\ \dfrac{k_z}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \end{pmatrix}$$

Linear wave

Linear wave... small-amplitude wave

Decomposition of a wave: 

Equilibrium part (time-independent) + Perturbed part (time-dependent)

$\rho_0(x, y, z)$ $\boldsymbol{v}_0(x, y, z) \equiv \mathbf{0}$ $\rho_1(x, y, z, t)$ $\boldsymbol{v}_1(x, y, z, t)$

$p_0(x, y, z)$ $\boldsymbol{B}_0(x, y, z)$ $p_1(x, y, z, t)$ $\boldsymbol{B}_1(x, y, z, t)$

$$\begin{aligned} \rho(x, y, z, t) & \qquad & \rho_0(x, y, z) + \rho_1(x, y, z, t) \\ \boldsymbol{v}(x, y, z, t) & \Rightarrow & \mathbf{0} + \boldsymbol{v}_1(x, y, z, t) \\ p(x, y, z, t) & & p_0(x, y, z) + p_1(x, y, z, t) \\ \boldsymbol{B}(x, y, z, t) & & \boldsymbol{B}_0(x, y, z) + \boldsymbol{B}_1(x, y, z, t) \end{aligned}$$

Perturbed part... **much smaller than** equilibrium part **($\square_0 \gg \square_1$)**

=> product of perturbed parts is **neglected**

$\square_1 \times \bigcirc_1$... 2nd-order $\fallingdotseq 0$

Linear approximation

Linearized MHD equations for linear waves

Assumption:

Equilibrium part

$$\rho_0(x,y,z)=\rho_0$$
$$p_0(x,y,z)=p_0$$
$$\boldsymbol{B}_0(x,y,z)=\boldsymbol{B}_0$$

uniform... Plasma and magnetic field are uniformly distributed.

Substitute the decomposed form into ideal MHD equations (no diffusion):

$$\rho_0+\rho_1(x,y,z,t)$$
$$\boldsymbol{0}+\boldsymbol{v}_1(x,y,z,t)$$
$$p_0+p_1(x,y,z,t)$$
$$\boldsymbol{B}_0+\boldsymbol{B}_1(x,y,z,t)$$

$$\begin{cases} \dfrac{\partial \rho}{\partial t}+\nabla\cdot(\rho\,\boldsymbol{v})=0 \\ \rho\left(\dfrac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\,\boldsymbol{v}\right)=-\nabla p+\dfrac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} \\ \dfrac{\partial}{\partial t}\left(\dfrac{p}{\gamma-1}\right)+\nabla\cdot\left(\dfrac{p}{\gamma-1}\,\boldsymbol{v}\right)=-p\,\nabla\cdot\boldsymbol{v} \quad \text{... adiabatic case} \\ \dfrac{\partial \boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{v}\times\boldsymbol{B}) \quad \text{... diffusion-free case} \end{cases}$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right)=0$$

Confirm these are the same.

Entropy

$$s=c_V\ln\left(\frac{p}{\rho^{\gamma}}\right)+constant$$

is conserved in the adiabatic case.

1. $\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \, \boldsymbol{v}) = 0$

$\rho_0 + \rho_1 (x, y, z, t)$
$\mathbf{0} + \boldsymbol{v}_1 (x, y, z, t)$ → $\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \, \boldsymbol{v}) = 0$

$$\frac{\partial \left(\rho_0 + \rho_1 (x, y, z, t) \right)}{\partial t} + \nabla \bullet \left[\left(\rho_0 + \rho_1 (x, y, z, t) \right) \left(\mathbf{0} + \boldsymbol{v}_1 (x, y, z, t) \right) \right] = 0$$

$$\nabla \bullet \left(\rho_0 \, \boldsymbol{v}_1 (x, y, z, t) + \rho_1 (x, y, z, t) \, \boldsymbol{v}_1 (x, y, z, t) \right) \approx \nabla \bullet \left(\rho_0 \, \boldsymbol{v}_1 (x, y, z, t) \right)$$

2nd-order

$\square_1 \bigcirc_1 \fallingdotseq 0$

$\frac{\partial \rho_0}{\partial t} = 0$ (ρ_0 *is constant*)

(ρ_0 *is constant*)

$$\frac{\partial \, \rho_1 (x, y, z, t)}{\partial t}$$

$$\rho_0 \nabla \bullet \boldsymbol{v}_1 (x, y, z, t)$$

Linearized mass conservation equation

$$\frac{\partial \, \rho_1 (x, y, z, t)}{\partial t} = - \rho_0 \nabla \bullet \boldsymbol{v}_1 (x, y, z, t)$$

$$2.\quad \rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\bullet\nabla\,\boldsymbol{v}\right) = -\nabla p + \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$$

$\rho_0 + \rho_1(x, y, z, t)$

$\mathbf{0} + \boldsymbol{v}_1(x, y, z, t)$

$p_0 + p_1(x, y, z, t)$

$\boldsymbol{B}_0 + \boldsymbol{B}_1(x, y, z, t)$

$$\longrightarrow \rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\bullet\nabla\,\boldsymbol{v}\right) = -\nabla p + \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$$

Left-hand side:

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\bullet\nabla\,\boldsymbol{v}\right) \longrightarrow \left(\rho_0 + \rho_1(x, y, z, t)\right)\left[\frac{\partial \boldsymbol{v}_1(x, y, z, t)}{\partial t} + \left(\boldsymbol{v}_1(x, y, z, t)\right)\bullet\nabla\,\boldsymbol{v}_1(x, y, z, t)\right]$$

2nd-order

$\square_1\ \bigcirc_1 \fallingdotseq 0$

2nd-order

$\square_1\ \bigcirc_1 \fallingdotseq 0$

$$\longrightarrow \rho_0\,\frac{\partial \boldsymbol{v}_1(x, y, z, t)}{\partial t}$$

Right-hand side:

$$-\nabla p + \frac{1}{\mu}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

$$\rightarrow -\nabla(p_0 + p_1(x,y,z,t)) + \frac{1}{\mu_0}[\nabla \times (\boldsymbol{B_0} + \boldsymbol{B_1}(x,y,z,t))] \times (\boldsymbol{B_0} + \boldsymbol{B_1}(x,y,z,t))$$

$(p_0 \; is \; constant)$ $(\boldsymbol{B}_0 \; is \; constant)$

2nd-orde

$\square_1 \bigcirc_1 \fallingdotseq 0$

$$\rightarrow -\nabla p_1(x,y,z,t) + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B_1}(x,y,z,t)) \times \boldsymbol{B_0}$$

Left-hand side

Right-hand side

$$\rho_0 \frac{\partial \boldsymbol{v_1}(x,y,z,t)}{\partial t} = -\nabla p_1(x,y,z,t) + \frac{1}{\mu_0}(\nabla \times \boldsymbol{B_1}(x,y,z,t)) \times \boldsymbol{B_0}$$

Linearized momentum equation