Role of Magnetic Field

It drives evolution of flow velocity.

$$\rho \frac{d\boldsymbol{v}}{dt} = \frac{1}{\mu_0} \left(\nabla \times \boldsymbol{B} \right) \times \boldsymbol{B}$$

Magnetic field in MHD

=> two components of ***j***×***B*** force (**magnetic tension** & **magnetic pressure**)

Decompose $\boldsymbol{j} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$ into two components:

A formula of vector differentiation:

$$\nabla(\boldsymbol{X} \bullet \boldsymbol{Y}) = (\boldsymbol{Y} \bullet \nabla)\boldsymbol{X} + (\boldsymbol{X} \bullet \nabla)\boldsymbol{Y} + \boldsymbol{Y} \times (\nabla \times \boldsymbol{X}) + \boldsymbol{X} \times (\nabla \times \boldsymbol{Y})$$

Substitute **X = B, Y = B** into this formula, then we have

$$\nabla(\boldsymbol{B} \bullet \boldsymbol{B}) = (\boldsymbol{B} \bullet \nabla)\boldsymbol{B} + (\boldsymbol{B} \bullet \nabla)\boldsymbol{B} + \boldsymbol{B} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{B})$$

This is transformed into

$$(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} = (\boldsymbol{B} \bullet \nabla)\boldsymbol{B} - \nabla\left(\frac{B^2}{2}\right)$$

$$\boldsymbol{j} \times \boldsymbol{B} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

$$= \underbrace{\frac{1}{\mu_0}(\boldsymbol{B} \bullet \nabla)\boldsymbol{B}}_{\text{tension part}} \underbrace{- \nabla\left(\frac{B^2}{2\mu_0}\right)}_{\text{pressure part}}$$

Magnetic tension: $\boldsymbol{F}_T = \frac{1}{\mu_0}(\boldsymbol{B}\bullet\nabla)\boldsymbol{B}$

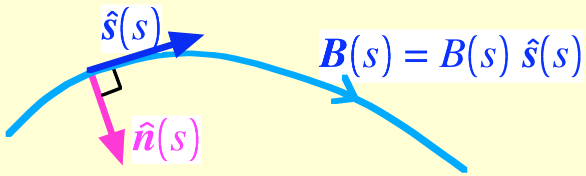
$$(\boldsymbol{B}\bullet\nabla)\boldsymbol{B} = B\,\hat{\boldsymbol{s}}\bullet\nabla(B\,\hat{\boldsymbol{s}})$$

$$= B\frac{\partial}{\partial s}(B\,\hat{\boldsymbol{s}})$$

$$= \left(B\frac{\partial B}{\partial s}\right)\hat{\boldsymbol{s}} + B^2\frac{\partial\hat{\boldsymbol{s}}}{\partial s}$$

$$= \frac{\partial}{\partial s}\left(\frac{1}{2}B^2\right)\hat{\boldsymbol{s}} + B^2\frac{\hat{\boldsymbol{n}}}{R_c}$$

$\hat{\boldsymbol{s}}(s)$… tangential vector along a magnetic field line (unit vector)



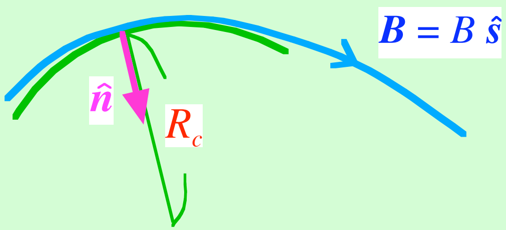
$$\hat{s}\bullet\frac{\partial\hat{s}}{\partial s} = \frac{1}{2}\frac{\partial}{\partial s}(\hat{s}\bullet\hat{s}) = \frac{1}{2}\frac{\partial}{\partial s}(1) = 0$$

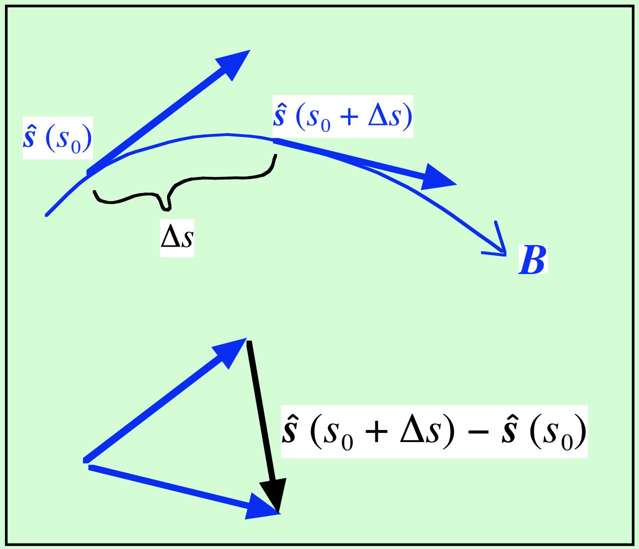
$\Rightarrow \hat{s}\perp\frac{\partial\hat{s}}{\partial s} \Rightarrow \frac{\partial\hat{s}}{\partial s} \mathbin{//} \hat{n}$ normal vector (unit vector)

$$\frac{\partial\hat{\boldsymbol{s}}}{\partial s} = \lim_{\Delta s\to 0}\frac{\hat{\boldsymbol{s}}(s_0+\Delta s) - \hat{\boldsymbol{s}}(s_0)}{\Delta s}$$

$$= \kappa\,\hat{\boldsymbol{n}} = \frac{\hat{\boldsymbol{n}}}{R_c}$$

κ… *curvature*, $R_c \equiv \frac{1}{\kappa}$… *curvature radius*



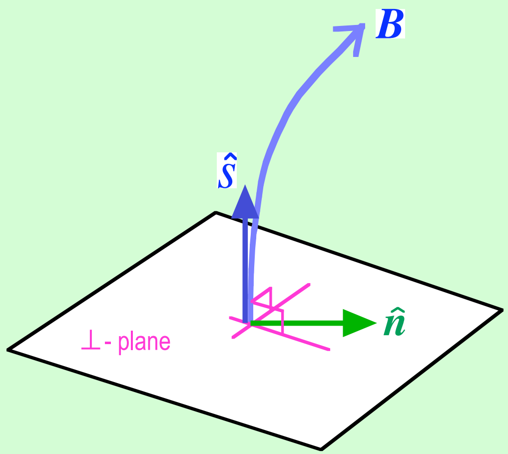


$$\boldsymbol{j} \times \boldsymbol{B} = \frac{1}{\mu_0} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

$$= \frac{1}{\mu_0} (\boldsymbol{B} \bullet \nabla) \boldsymbol{B} - \nabla \left(\frac{B^2}{2\mu_0} \right)$$

$$\frac{\partial}{\partial s} \left(\frac{B^2}{2\mu_0} \right) \hat{\boldsymbol{s}} + \frac{B^2}{\mu_0} \frac{\hat{\boldsymbol{n}}}{R_c} \quad - \left[\frac{\partial}{\partial s} \left(\frac{B^2}{2\mu_0} \right) \hat{\boldsymbol{s}} + \nabla_\perp \left(\frac{B^2}{2\mu_0} \right) \right]$$

canceled



$$\boldsymbol{j} \times \boldsymbol{B} = \frac{B^2}{\mu_0} \frac{\hat{\boldsymbol{n}}}{R_c} - \nabla_\perp \left(\frac{B^2}{2\mu_0} \right)$$

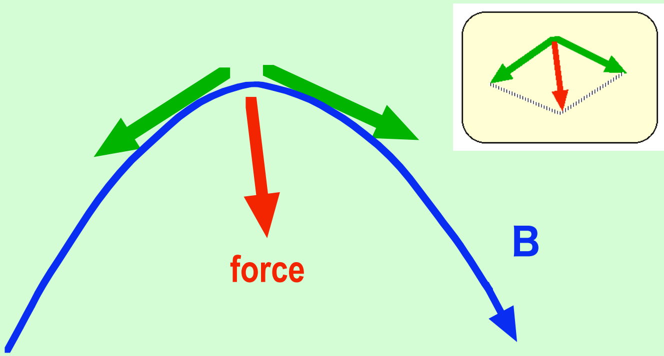
magnetic tension force | magnetic pressure gradient force

Both **magnetic tension force** and **magnetic pressure gradient force** work in the direction **perpendicular to magnetic field**.

Features of magnetic tension and magnetic pressure

Magnetic tension force

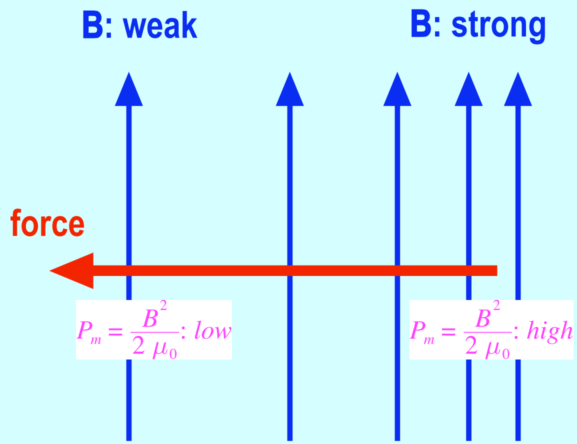
... depends on **shape of a field line**



$$\frac{B^2}{\mu_0}\frac{\hat{\boldsymbol{n}}}{R_c}$$

Magnetic pressure gradient force

... depends on **distribution of field strength**

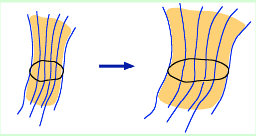


$$-\nabla_\perp\left(\frac{B^2}{2\mu_0}\right)$$

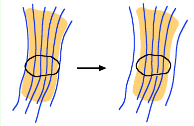
Summary of the roles of E^{MHD} and B in MHD

Electric field => drives evolution of magnetic field through $\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}^{MHD}$

$$-\nabla \times \boldsymbol{E}_{conv} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$$

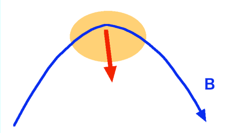


$$-\nabla \times \boldsymbol{E}_{resis} = -\nabla \times (\eta_{diff} \nabla \times \boldsymbol{B})$$

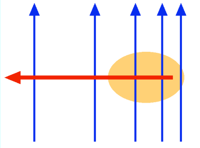


Magnetic Field => drives evolution of flow velocity through $\rho \frac{d\boldsymbol{v}}{dt} = \frac{1}{\mu_0} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$

$$\boldsymbol{F}_t = \frac{B^2}{\mu_0} \frac{\hat{\boldsymbol{n}}}{R_c}$$



$$\boldsymbol{F}_m = -\nabla_{\perp} \left(\frac{B^2}{2\mu_0} \right)$$



MHD Wave

What is a wave?

It is a time-dependent phenomenon in which a **physical state** is transported from one location to other locations through a medium (what is transported is **physical state**, not physical object).