Roles of Electric Field & Magnetic Field in MHD

Role of Electric Field

It drives evolution of magnetic field.

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}^{MHD}$$

Electric field in MHD => two major components

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}^{MHD} = -\nabla \times (-\boldsymbol{v} \times \boldsymbol{B}) - \nabla \times (\eta_{diff} \nabla \times \boldsymbol{B})$$

convective term

$$\boldsymbol{E}_{conv} \equiv -\boldsymbol{v} \times \boldsymbol{B}$$

diffusion term

$$\boldsymbol{E}_{resis} \equiv \eta_{diff} \nabla \times \boldsymbol{B}$$

- **Convective term:** $-\nabla \times \boldsymbol{E}_{conv} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$

  it drives evolution of magnetic field **via** macroscale motion (flow) of fluid elements.

- **Diffusion term:** $-\nabla \times \boldsymbol{E}_{resis} = -\nabla \times (\eta_{diff} \nabla \times \boldsymbol{B})$

  it drives evolution of magnetic field **via** microscale motion (collision) of particles.

Convective term: $-\nabla \times \boldsymbol{E}_{conv} = \nabla \times \left(\boldsymbol{v} \times \boldsymbol{B}\right)$

Convective limit...

$$\frac{\partial \boldsymbol{B}'}{\partial t'} = \nabla' \times \left(\boldsymbol{v}' \times \boldsymbol{B}' - \frac{1}{Rm} \nabla' \times \boldsymbol{B}' \right)$$

... dimensionless form

$$Rm \equiv \frac{v_0 \, l_0}{\eta_{diff}} >> 1$$

→ **we can neglect diffusion**

Three conditions for convective limit:

- **I. typical length** is **very large**
- **II. typical velocity** is **very large**
- **III. magnetic diffusivity** is **small**

e.g. solar corona...

condition I: $l_0 \sim 10^7$ m

condition II: $v_0 \sim 10^5$ m/s

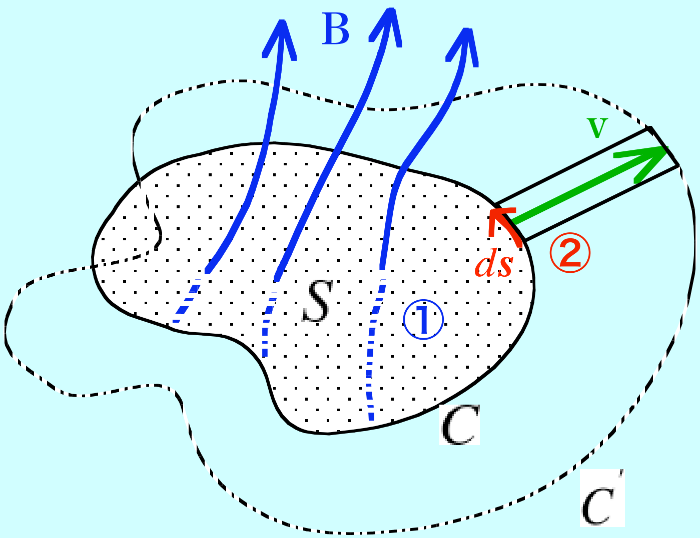
condition III: $\eta_{diff} = 5.2 \times 10^7 \dfrac{\ln N_D}{T^{3/2}} \sim 0.1$ (m^2/s)

→ $Rm \sim 10^{13} >> 1$

Evolution of magnetic field via $\boldsymbol{E}_{conv} \equiv -\boldsymbol{v} \times \boldsymbol{B}$:

$$\Phi_S = \iint_S \boldsymbol{B} \bullet d\boldsymbol{S}$$

... total magnetic flux crossing S



Lagrangian derivative

$$\left\{ \begin{array}{c} \textit{Change of } B \textit{ when } C \textit{ is fixed} \; ① \\ + \\ \textit{Change of } C \textit{ when } B \textit{ is fixed} \; ② \end{array} \right.$$

$$\frac{d\Phi_S}{dt} = \underbrace{\iint_S \frac{\partial \boldsymbol{B}}{\partial t} \bullet d\boldsymbol{S}}_{①} + \underbrace{\oint_C \boldsymbol{B} \bullet (\boldsymbol{v} \times d\boldsymbol{s})}_{②}$$

$$= \iint_S \frac{\partial \boldsymbol{B}}{\partial t} \bullet d\boldsymbol{S} + \oint_C (\boldsymbol{B} \times \boldsymbol{v}) \bullet d\boldsymbol{s}$$

scalar triple vector product

$$= \iint_S \frac{\partial \boldsymbol{B}}{\partial t} \bullet d\boldsymbol{S} + \iint_S \nabla \times (\boldsymbol{B} \times \boldsymbol{v}) \bullet d\boldsymbol{S}$$

Stokes's theorem

$$= \iint_S \left[\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{B} \times \boldsymbol{v}) \right] \bullet d\boldsymbol{S}$$

$$= \iint_S \left[\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \right] \bullet d\boldsymbol{S}$$

$$= 0$$

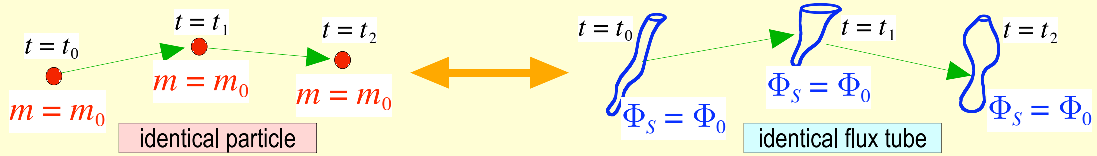
$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \Rightarrow \frac{d\Phi_S}{dt} = 0$$

Physical meaning of $\frac{d\Phi_S}{dt} = 0$ => **Frozen-in evolution**

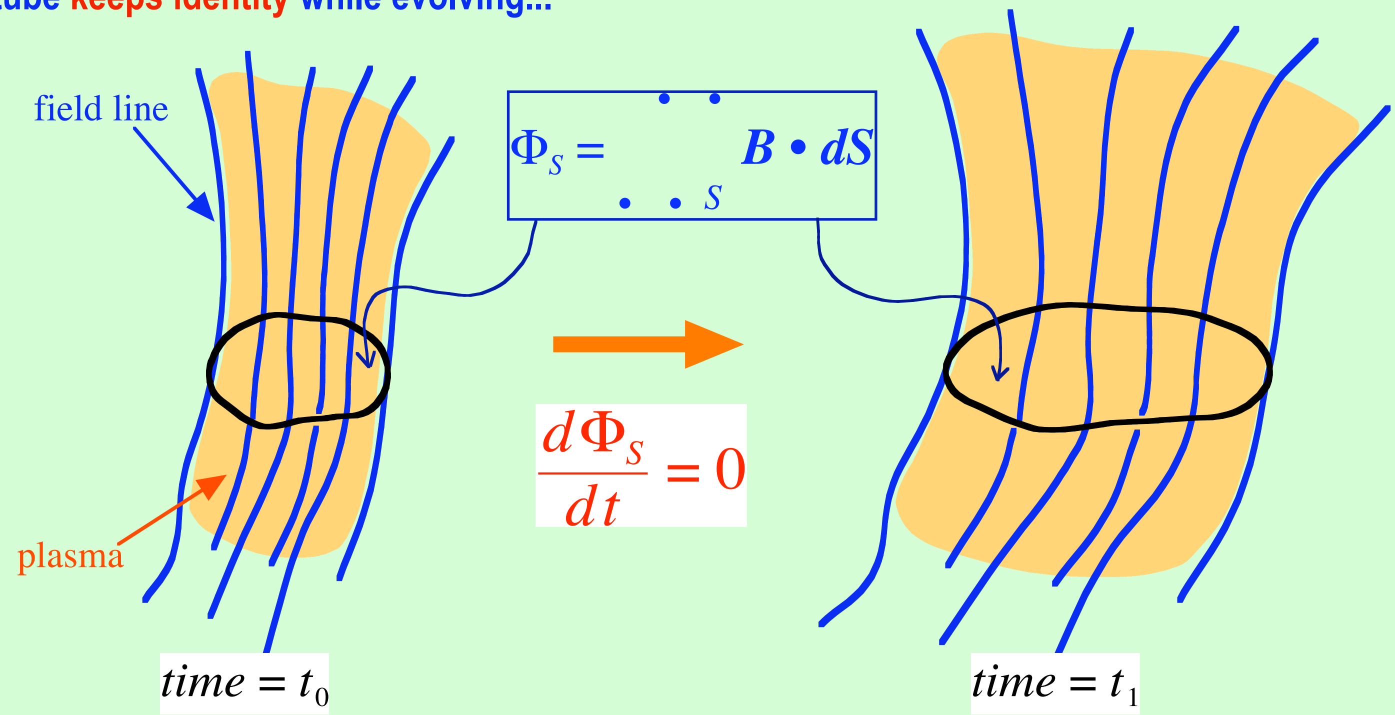
The **number of magnetic field lines** passing through the area bounded by any closed curve that moves together with a plasma **does not change with time**.

⇓

We can determine an **identical flux tube** in which magnetic field lines **are frozen into** the plasma.



Flux tube keeps identity while evolving...



Diffusion term: $-\nabla \times \boldsymbol{E}_{resis} = -\nabla \times \left(\eta_{diff} \nabla \times \boldsymbol{B}\right)$

Diffusive limit...

$$\frac{\partial \boldsymbol{B}'}{\partial t'} = \nabla' \times \left(\boldsymbol{v}' \times \boldsymbol{B}' - \frac{1}{Rm} \nabla' \times \boldsymbol{B}' \right)$$

$$Rm \equiv \frac{v_0 \, l_0}{\eta_{diff}} \lesssim 1$$ → **we can<u>not</u> neglect diffusion**

Three conditions for diffusive limit:

- **I. typical length** is **very small**
- **II. typical velocity** is **very small**
- **III. magnetic diffusivity** is **large**

In astronomical plasmas, their typical length l_0 is usually very large, so $Rm \equiv \frac{v_0 \, l_0}{\eta_{diff}} \lesssim 1$ is rare except for a region where magnetic field changes sharply (e.g. current sheet), which significantly reduces l_0.

It may also enhance η_{diff} via kinetic processes (e.g. wave-particle interaction).

Evolution of magnetic field via $\boldsymbol{E}_{resis} \equiv \eta_{diff} \nabla \times \boldsymbol{B}$:

When η_{diff} is constant, $\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times (\eta_{diff} \nabla \times \boldsymbol{B}) \Rightarrow \frac{\partial \boldsymbol{B}}{\partial t} = \eta_{diff} \nabla^2 \boldsymbol{B}$

$$\frac{\partial q}{\partial t} = \eta_{diff} \nabla^2 q$$ **... diffusion equation**

$q = T, j_x, B_y, \ldots$

Time scale of diffusion: τ_{diff}

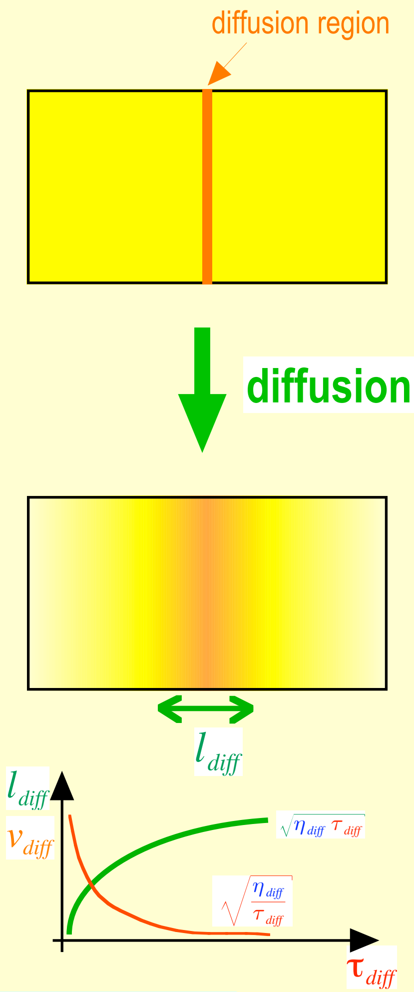
$$\frac{\partial q}{\partial t} = \eta_{diff} \nabla^2 q \Rightarrow \frac{q}{\tau_{diff}} \sim \eta_{diff} \frac{q}{l_{diff}^2} \rightarrow \tau_{diff} \sim \frac{l_{diff}^2}{\eta_{diff}}$$

l_{diff}: *width of diffusion region*

Length scale of diffusion: $l_{diff} \sim \sqrt{\eta_{diff} \tau_{diff}}$

Velocity scale of diffusion: v_{diff}

$$v_{diff} \sim \frac{l_{diff}}{\tau_{diff}} = \frac{\sqrt{\eta_{diff} \tau_{diff}}}{\tau_{diff}} = \sqrt{\frac{\eta_{diff}}{\tau_{diff}}}$$



Diffusion proceeds fast at the beginning and then slowly.

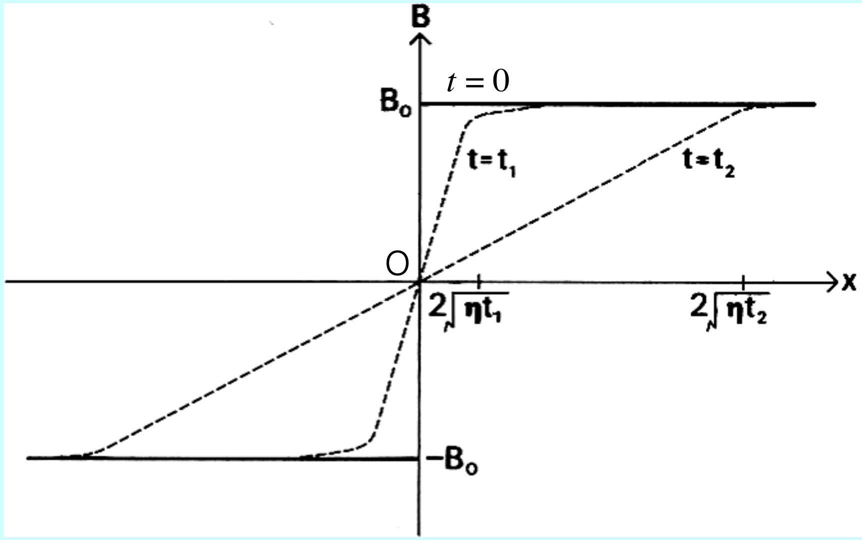
Diffusion of an antiparallel magnetic field (annihilation)

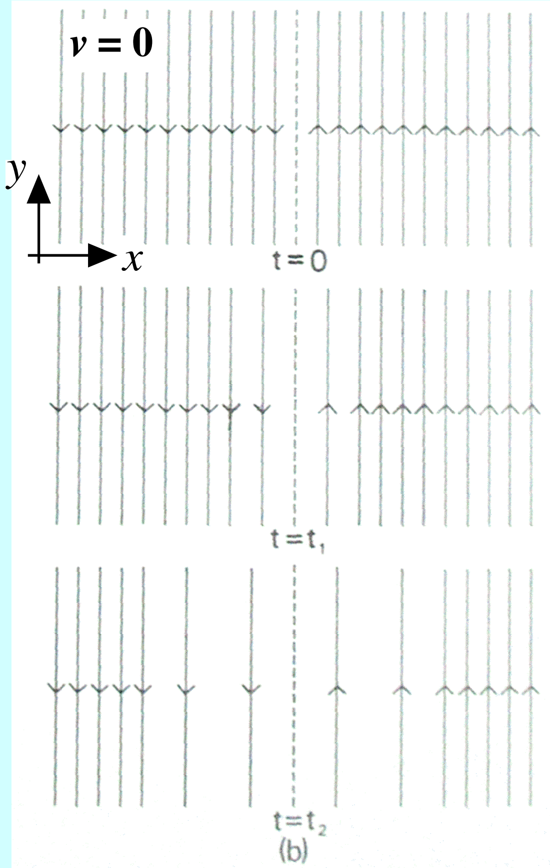
$$\frac{\partial B_y}{\partial t} = \eta_{diff} \frac{\partial^2}{\partial x^2} B_y$$

$B_y(x, t=0) = \begin{cases} B_0 & for\ x>0 \\ -B_0 & for\ x<0 \end{cases}$ ... initial condition

$B_y(x=\pm\infty, t) = \begin{cases} B_0 & for\ x=\infty \\ -B_0 & for\ x=-\infty \end{cases}$ ... boundary condition

$$B_y(x,t) = \frac{2B_0}{\sqrt{\pi}} \operatorname{erf}\left(\frac{x}{\sqrt{4\eta_{diff}\, t}}\right), \operatorname{erf}(\xi) = \int_0^{\xi} e^{-u^2} du$$





Solar MHD (Priest 1982)

- magnetic field is annihilated at x = 0
- magnetic field diffuses through a plasma => **violates frozen-in evolution**