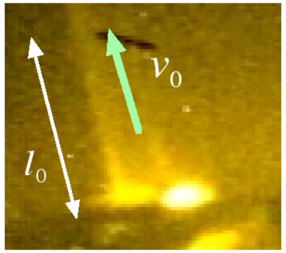
Dimensionless form of MHD equations

Normalization based on typical scale of a phenomenon



Typical scale:

length… l_0

velocity… v_0

time… $t_0 = \frac{l_0}{v_0}$

Typical scale of a phenomenon ($l_0, v_0, P_0, \rho_0, B_0$)

→ Unit of physical quantity

e.g. velocity

dimensionless value (e.g. 2.5)

$$v' = \frac{v}{v_0}$$

v → real value (e.g. 5×10^4 m / s)

v_0 → typical scale (e.g. 2×10^4 m / s)

Unit of velocity

Dimensionless quantities:

$$l' = \frac{l}{l_0} \qquad \boldsymbol{v}' = \frac{\boldsymbol{v}}{v_0} \qquad t' = \frac{t}{t_0}$$

$$\rho' = \frac{\rho}{\rho_0} \qquad P' = \frac{P}{P_0} \qquad T' = \frac{T}{T_0}$$

$$\boldsymbol{B}' = \frac{\boldsymbol{B}}{B_0}$$

We then derive the **dimensionless form** of MHD equations.

Mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho\, \boldsymbol{v}) = 0$$

$$\rho = \rho' \rho_0$$

$$t = t' t_0$$

$$\boldsymbol{v} = \boldsymbol{v}' v_0$$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial [x' l_0]} \\ \frac{\partial}{\partial [y' l_0]} \\ \frac{\partial}{\partial [z' l_0]} \end{pmatrix} = \frac{1}{l_0} \begin{pmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \\ \frac{\partial}{\partial z'} \end{pmatrix} = \frac{1}{l_0} \nabla'$$

$$\frac{\partial [\rho' \rho_0]}{\partial [t' t_0]} + \left[\frac{1}{l_0} \nabla'\right] \bullet \left([\rho' \rho_0][\boldsymbol{v}' v_0]\right) = 0$$

$$\frac{\rho_0}{t_0} \frac{\partial \rho'}{\partial t'} + \frac{\rho_0 v_0}{l_0} \nabla' \bullet (\rho' \boldsymbol{v}') = 0$$

$$\frac{\rho_0}{t_0} = \frac{\rho_0}{\left(\frac{l_0}{v_0}\right)} = \frac{\rho_0 v_0}{l_0}$$

$$t_0 = \frac{l_0}{v_0}$$

$$\frac{\partial \rho'}{\partial t'} + \nabla' \bullet (\rho' \boldsymbol{v}') = 0$$

Momentum equation:

$$\rho \frac{d\boldsymbol{v}}{dt} = -\nabla p + \boldsymbol{j}\times\boldsymbol{B} \Rightarrow \rho \frac{d\boldsymbol{v}}{dt} = -\nabla p + \frac{1}{\mu_0}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$$

$$\frac{\rho_0 v_0}{t_0}\rho' \frac{d\boldsymbol{v}'}{dt'} = -\frac{p_0}{l_0}\nabla' p' + \frac{B_0^2}{l_0}\frac{1}{\mu_0}\left(\nabla'\times\boldsymbol{B}'\right)\times\boldsymbol{B}'$$

$$t_0 = \frac{l_0}{v_0}$$

$$v_0^2\rho' \frac{d\boldsymbol{v}'}{dt'} = -\frac{p_0}{\rho_0}\nabla' p' + \frac{B_0^2}{\rho_0}\frac{1}{\mu_0}\left(\nabla'\times\boldsymbol{B}'\right)\times\boldsymbol{B}'$$

$c_{s0} \equiv \sqrt{\frac{p_0}{\rho_0}}$ ... isothermal sound speed

$v_{A0} \equiv \frac{B_0}{\sqrt{\mu_0\rho_0}}$ ... Alfvén speed

$$\rho' \frac{d\boldsymbol{v}'}{dt'} = -\frac{c_{s0}^2}{v_0^2}\nabla' p' + \frac{v_{A0}^2}{v_0^2}\left(\nabla'\times\boldsymbol{B}'\right)\times\boldsymbol{B}'$$

Energy equation:

$$\rho \frac{d}{dt}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right) + p\,\nabla \bullet \boldsymbol{v} = \eta\, j^2 \;\rightarrow\; \eta \left| \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \right|^2 = \eta_{diff} \frac{|\nabla \times \boldsymbol{B}|^2}{\mu_0}$$

$$\frac{\rho_0\, p_0}{t_0\, \rho_0}\rho' \frac{d}{dt'}\left(\frac{1}{\gamma - 1}\frac{p'}{\rho'}\right) + \frac{p_0\, v_0}{l_0} p'\,\nabla' \bullet \boldsymbol{v}' = \frac{B_0^2}{l_0^2}\eta_{diff}\frac{\left|\nabla' \times \boldsymbol{B}'\right|^2}{\mu_0}$$

$$t_0 = \frac{l_0}{v_0}$$

$$\rho' \frac{d}{dt'}\left(\frac{1}{\gamma - 1}\frac{p'}{\rho'}\right) + p'\,\nabla' \bullet \boldsymbol{v}' = \frac{\frac{B_0^2}{\mu_0}}{p_0}\frac{\eta_{diff}}{v_0\, l_0}\left|\nabla' \times \boldsymbol{B}'\right|^2$$

$$\rho' \frac{d}{dt'}\left(\frac{1}{\gamma - 1}\frac{p'}{\rho'}\right) + p'\,\nabla' \bullet \boldsymbol{v}' = \frac{2}{\beta}\frac{1}{\mathrm{Rm}}\left|\nabla' \times \boldsymbol{B}'\right|^2$$

Dimensionless parameters

$$\beta \equiv \frac{p_0}{p_{m0}} = 2\frac{c_{s0}^2}{v_{A0}^2} \quad \left(p_{m0} \equiv \frac{B_0^2}{2\,\mu_0}\text{: magnetic pressure}\right)$$

... plasma beta

$$\mathrm{Rm} \equiv \frac{v_0\, l_0}{\eta_{diff}}$$

... magnetic Reynolds number

Equation of state:

$$p = \rho \frac{k_B}{\bar{m}} T$$

$$\downarrow$$

$$p_0 p' = \rho_0 \rho' \frac{k_B}{\bar{m}} T_0 T'$$

$$\downarrow$$

$$p_0 p' = \rho_0 \frac{k_B}{\bar{m}} T_0 \rho' T'$$

$$T_0 = \frac{p_0}{\rho_0} \frac{\bar{m}}{k_B} \rightarrow \downarrow$$

$$p' = \rho' T'$$

MHD induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left(\boldsymbol{v} \times \boldsymbol{B} - \eta_{diff} \nabla \times \boldsymbol{B} \right)$$

$$\frac{B_0}{t_0} \frac{\partial \boldsymbol{B}'}{\partial t'} = \frac{1}{l_0} \nabla' \times \left(v_0 B_0 \boldsymbol{v}' \times \boldsymbol{B}' - \eta_{diff} \frac{B_0}{l_0} \nabla' \times \boldsymbol{B}' \right)$$

$$t_0 = \frac{l_0}{v_0}$$

$$\frac{\partial \boldsymbol{B}'}{\partial t'} = \nabla' \times \left(\boldsymbol{v}' \times \boldsymbol{B}' - \frac{\eta_{diff}}{v_0 l_0} \nabla' \times \boldsymbol{B}' \right)$$

$$\mathrm{Rm} \equiv \frac{v_0 l_0}{\eta_{diff}}$$

$$\frac{\partial \boldsymbol{B}'}{\partial t'} = \nabla' \times \left(\boldsymbol{v}' \times \boldsymbol{B}' - \frac{1}{\mathrm{Rm}} \nabla' \times \boldsymbol{B}' \right)$$