per unit volume

$$\frac{1}{V} T \frac{dS}{dt} - \frac{1}{V} p \frac{dV}{dt} = \frac{1}{V} \frac{dU}{dt}$$

$$\frac{1}{V}\frac{dU}{dt} \rightarrow \rho \frac{d}{dt}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right)$$

$$\frac{1}{V} p \frac{dV}{dt} \rightarrow p \nabla \cdot \boldsymbol{v}$$

$$\frac{1}{V} T \frac{dS}{dt} \rightarrow -\nabla \cdot \boldsymbol{F}_c + \eta j^2 + H$$

Internal energy eq. in MHD

$$\rho \frac{d}{dt}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right) = -p \nabla \cdot \boldsymbol{v} - \nabla \cdot \boldsymbol{F}_c + \eta j^2 + H$$

+ radiation (flux)

- $-p \nabla \cdot \boldsymbol{v}$: **compression** (source; reversible)
- $-\nabla \cdot \boldsymbol{F}_c$: **conduction** (flux)
- $\eta j^2 + H$: **heating** (source; irreversible)

$$= \rho \frac{\partial}{\partial t}\left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right) + \rho \boldsymbol{v} \cdot \nabla \left(\frac{1}{\gamma - 1}\frac{p}{\rho}\right) = \frac{\partial}{\partial t}\left(\frac{p}{\gamma - 1}\right) + \nabla \cdot \left(\frac{p}{\gamma - 1}\boldsymbol{v}\right)$$

$\nabla \cdot \left(\frac{p}{\gamma - 1}\boldsymbol{v}\right)$: **convection** (flux)

Equation of state (fully ionized)

$P^{p} = n\, k_B\, T^{p}$ ... proton's equation of state

$P^{e} = n\, k_B\, T^{e}$ ... electron's equation of state

summation

$$P^{MHD} = 2\, n\, k_B\, T^{MHD} = 2\, \frac{\rho^{MHD}}{M+m}\, k_B\, T^{MHD} = \rho^{MHD}\, \frac{k_B}{\frac{M+m}{2}}\, T^{MHD} = \rho^{MHD}\, \frac{k_B}{\bar{m}}\, T^{MHD}$$

$T^{p} \sim T^{e} \sim T^{MHD}$

$n = \frac{\rho^{MHD}}{M+m}$

$\bar{m} \equiv \frac{M+m}{2}$

mean mass

$$P^{MHD} = \rho^{MHD}\, \frac{k_B}{\bar{m}}\, T^{MHD}$$

Equation of state in MHD

Charge conservation equation

$\frac{e}{M} \times$ $\frac{\partial \rho^p}{\partial t} + \nabla \bullet (\rho^p \boldsymbol{v}^p) = 0$ ... proton's mass conservation $\rho^p \equiv n_p M$

$\frac{e}{m} \times$ $\frac{\partial \rho^e}{\partial t} + \nabla \bullet (\rho^e \boldsymbol{v}^e) = 0$ ... electron's mass conservation $\rho^e \equiv n_e m$

subtraction

$$\frac{\partial \rho_c}{\partial t} + \nabla \bullet \boldsymbol{j} = 0$$

$\rho_c \equiv e(n_p - n_e)$

$\boldsymbol{j} \equiv e(n_p \boldsymbol{v}^p - n_e \boldsymbol{v}^e)$

Charge conservation equation

In MHD, $n_p \sim n_e \sim n$, so $\rho_c^{MHD} \sim 0$

$$\nabla \bullet \boldsymbol{j}^{MHD} \sim 0$$

$\boldsymbol{j}^{MHD} \equiv ne(\boldsymbol{v}^p - \boldsymbol{v}^e)$

Charge conservation equation in MHD

$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}^{MHD}$ is consistent with this equation.

MHD equations

Fluid part

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$ ... mass conservation

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \right) = - \nabla p + \boldsymbol{j} \times \boldsymbol{B} + \boldsymbol{F}$$ ... momentum equation

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla$$

$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} \right) + \nabla \cdot \left(\frac{p}{\gamma - 1} \boldsymbol{v} \right) = - p \nabla \cdot \boldsymbol{v} - \nabla \cdot \boldsymbol{F}_c + \eta j^2$$ ... internal energy equation

$$\boldsymbol{F}_c = - \kappa_c \nabla T$$

$$p = \rho \frac{k_B}{m} T$$ ... equation of state

Electromagnetic part

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left(\boldsymbol{v} \times \boldsymbol{B} - \eta_{diff} \nabla \times \boldsymbol{B} \right)$$ ... induction equation

$$\nabla \cdot \boldsymbol{B} = 0$$ ... magnetic flux conservation (initial condition)

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$ ... Ampere's law

$$\boldsymbol{E} = \eta \boldsymbol{j} - \boldsymbol{v} \times \boldsymbol{B}$$ ... Ohm's law

We have 9 quantities: $\rho, v_x, v_y, v_z, P, T, B_x, B_y, B_z$

We have 9 equations:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho\, \boldsymbol{v}) = 0$$ ... for ρ

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \bullet \nabla\, \boldsymbol{v} \right) = -\nabla p + \frac{1}{\mu_0} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \boldsymbol{F}$$ ... for v_x, v_y, v_z

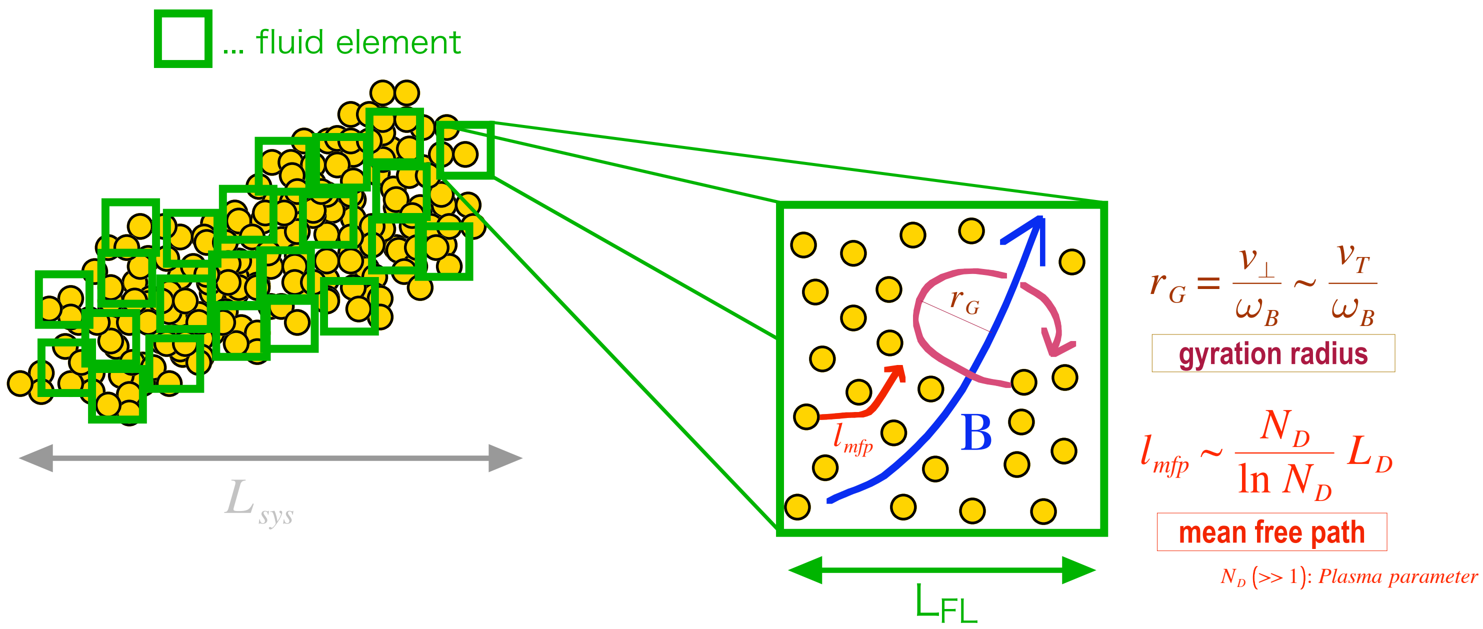
$$\frac{\partial}{\partial t} \left(\frac{p}{\gamma - 1} \right) + \nabla \bullet \left(\frac{p}{\gamma - 1}\, \boldsymbol{v} \right) = -p\, \nabla \bullet \boldsymbol{v} + \nabla \bullet (\kappa_c \nabla T) + \eta\, j^2$$ ... for P

$$p = \rho \frac{k_B}{\overline{m}} T$$ ... for T

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left(\boldsymbol{v} \times \boldsymbol{B} - \eta_{diff} \nabla \times \boldsymbol{B} \right)$$ ... for B_x, B_y, B_z

Summary of MHD

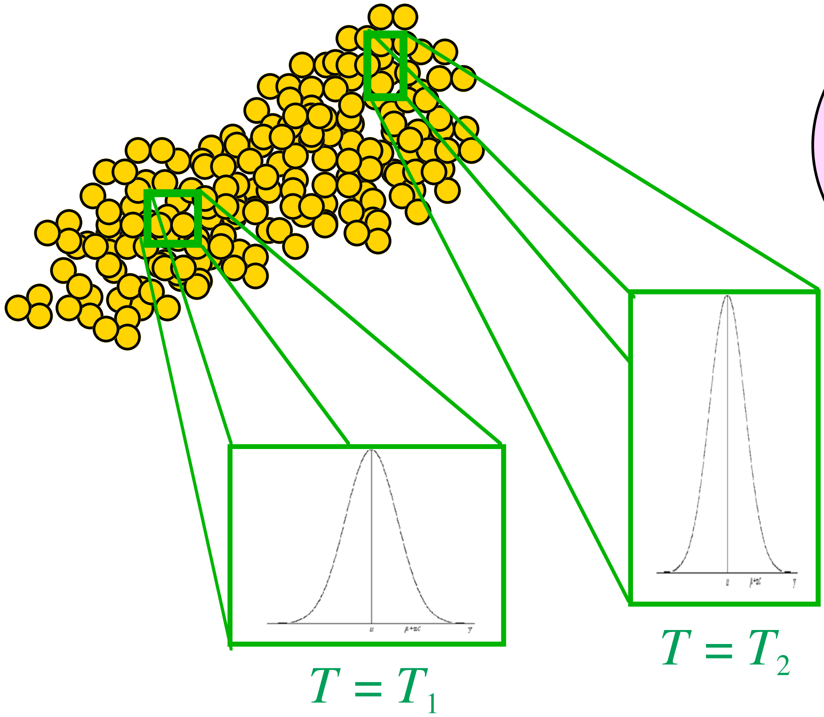
I. Fluid description



When typical length scale of a plasma system is much larger than mean free path and/or gyration radius, fluid element <u>**can**</u> be defined. => **Fluid approach**

When typical length scale of a plasma system is much smaller than mean free path and gyration radius, fluid element <u>**cannot**</u> be defined. => **Kinetic approach**

II. Local Thermal Equilibrium (LTE)



Maxwellian distribution

$$f(v_x) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}} \exp\left(-\frac{\frac{1}{2} m v_x^2}{k_B T}\right)$$

$$f(v_y) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}} \exp\left(-\frac{\frac{1}{2} m v_y^2}{k_B T}\right)$$

$$f(v_z) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{1}{2}} \exp\left(-\frac{\frac{1}{2} m v_z^2}{k_B T}\right)$$

※ in the case of photons => Plank distribution

$$n_{BE}(\nu) = \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}$$

In each fluid element, random velocity component of protons & electrons follows Maxwellian distribution characterized by a single physical quantity, temperature.

When temperature is different between fluid elements, it is called Local Thermal Equilibrium.

When temperature is the same between them, it is called Global Thermal Equilibrium.

III. Non-relativistic approximation

$$v_0 << c$$

IV. Single-fluid model

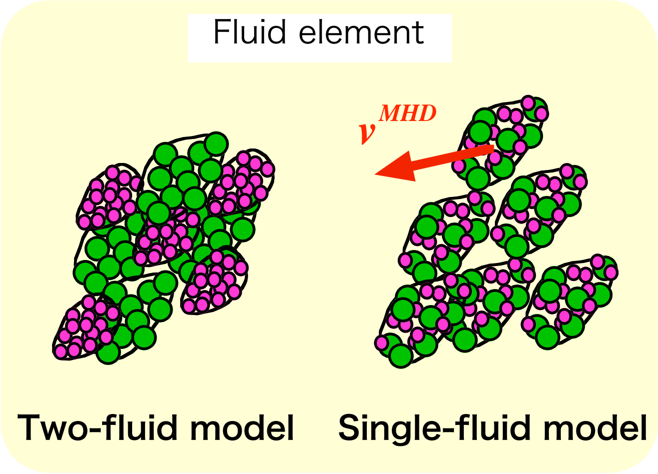
- **Local charge neutrality:**

$$n_p \sim n_e \sim n$$

- **Density of MHD fluid element** is close to **proton's density:**

$$\rho^{MHD} \equiv n\,M + n\,m = n\,M\left(1 + \frac{m}{M}\right) \approx n\,M$$

$\frac{1}{1830} << 1$



- **Velocity of MHD fluid element** is close to **proton's velocity:**

$$\boldsymbol{v}^{MHD} \equiv \frac{M\,\boldsymbol{v}^{p} + m\,\boldsymbol{v}^{e}}{M + m} \approx \boldsymbol{v}^{p}$$

- **Velocity of MHD fluid element in** $B_{\perp}\!-\!plane$ is given by **E_{conv} x B drift** (same for proton & electron)**:**

$$\frac{\boldsymbol{E}_{conv} \times \boldsymbol{B}}{B^2} = \frac{\left(-\boldsymbol{v}^{MHD} \times \boldsymbol{B}\right) \times \boldsymbol{B}}{B^2} = \frac{\left(-\boldsymbol{v}_{\perp}^{MHD} \times \boldsymbol{B}\right) \times \boldsymbol{B}}{B^2} = \boldsymbol{v}_{\perp}^{MHD}$$

V. Diffusion

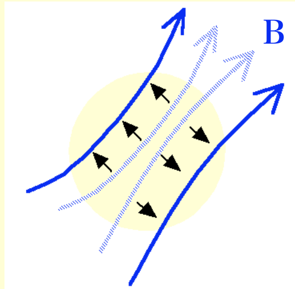
dimension of diffusivity = *velocity* (v) × *length* (l) ⇒ m^2/s

- **Magnetic diffusivity**

$$\eta_{diff} \equiv \frac{\eta}{\mu_0} = 5.2\times 10^{7} \frac{\ln N_D}{T^{3/2}} \ (\mathrm{m^2/s}) \propto T^{-3/2}$$

$\ln N_D \sim 10$ (Coulomb logarithm)

※ what is diffused is magnetic field.



- **Kinematic viscosity**

$$\nu = 2.21\times 10^{-16} \frac{T^{5/2}}{\rho \ln N_D} \ (\mathrm{m^2/s}) \propto T^{5/2} \rho^{-1}$$

※ what is diffused is proton.

- **Thermal diffusivity** $\alpha \equiv \dfrac{\kappa_C}{\rho\, c_p}$

$$\alpha_{//} = 4.3\times 10^{-16} \frac{T^{5/2}}{\rho \ln N_D} \ (\mathrm{m^2/s}) \propto T^{5/2} \rho^{-1}$$

$$\frac{\alpha_{\perp}}{\alpha_{//}} = 2.1\times 10^{-31} \frac{n^2}{T^3 B^2}$$

※ what is diffused is electron.