Lagrangian derivative vs. Eulerian derivative

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\boldsymbol{v} \bullet \nabla)$$

$$dA(x, y, z, t) = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz + \frac{\partial A}{\partial t} dt$$

total differential

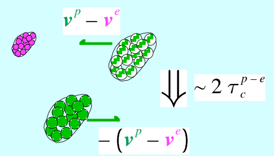
$$\begin{aligned}
\therefore \frac{dA}{dt} &= \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial z}\frac{dz}{dt} + \frac{\partial A}{\partial t}\frac{dt}{dt} \\
&= \frac{\partial A}{\partial x} v_x + \frac{\partial A}{\partial y} v_y + \frac{\partial A}{\partial z} v_z + \frac{\partial A}{\partial t} \\
&= \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right) A + \frac{\partial A}{\partial t} \\
&= \frac{\partial A}{\partial t} + (\boldsymbol{v} \bullet \nabla) A
\end{aligned}$$

Momentum equation in MHD (simplified version)

proton's momentum equation (nonlinear term $(\boldsymbol{v}^p \cdot \nabla)\boldsymbol{v}^p$ is neglected):

$$n\,M\,\frac{\partial \boldsymbol{v}^p}{\partial t} = -\nabla P^p + n\,e\left(\boldsymbol{E} + \boldsymbol{v}^p \times \boldsymbol{B}\right) + \boldsymbol{f}_c^{p-e}$$

$$\boldsymbol{f}_c^{p-e} \approx -n\,M\,\frac{\boldsymbol{v}^p - \boldsymbol{v}^e}{\tau_c^{p-e}}$$



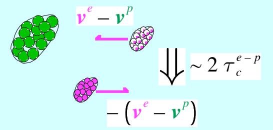
macroscale electric field (e.g. $-\boldsymbol{v}\times\boldsymbol{B}$)

collisional force (<= microscale electric field characterized by L_D)
= **change in momentum (relative velocity) per unit time**

electron's momentum equation (nonlinear term $(\boldsymbol{v}^e \cdot \nabla)\boldsymbol{v}^e$ is neglected):

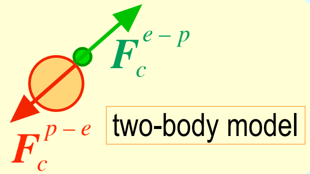
$$n\,m\,\frac{\partial \boldsymbol{v}^e}{\partial t} = -\nabla P^e - n\,e\left(\boldsymbol{E} + \boldsymbol{v}^e \times \boldsymbol{B}\right) + \boldsymbol{f}_c^{e-p}$$

$$\boldsymbol{f}_c^{e-p} \approx -n\,m\,\frac{\boldsymbol{v}^e - \boldsymbol{v}^p}{\tau_c^{e-p}}$$



Conservation of total momentum via collision

$$\boldsymbol{f}_c^{e-p} + \boldsymbol{f}_c^{p-e} = \boldsymbol{0}\ \left(\Rightarrow \tau_c^{e-p} = \frac{m}{M}\,\tau_c^{p-e}\right)$$



Add both equations...

$$n\,\frac{\partial}{\partial t}\left(M\,\boldsymbol{v}^p + m\,\boldsymbol{v}^e\right) = n\,e\left(\boldsymbol{v}^p - \boldsymbol{v}^e\right)\times\boldsymbol{B} - \nabla\left(P^p + P^e\right)$$

$$\rho^{MHD}\,\frac{\partial}{\partial t}\,\boldsymbol{v}^{MHD} = \boldsymbol{j}^{MHD}\times\boldsymbol{B} - \nabla P^{MHD}$$

Momentum eq. in MHD
(simplified version)

$$P^{MHD} \equiv P^p + P^e:\ \textit{total gas pressure}$$

$$\boldsymbol{j}^{MHD} \equiv n\,e\,\left(\boldsymbol{v}^p - \boldsymbol{v}^e\right):\ \textit{current density}$$

Nonlinear term of flow velocity: $(\boldsymbol{v} \bullet \nabla)\boldsymbol{v}$

$$(\rho^p v^p{}_k) \frac{\partial}{\partial x_k} v^p{}_i + (\rho^e v^e{}_k) \frac{\partial}{\partial x_k} v^e{}_i = n\, M \left(v^p{}_k \frac{\partial}{\partial x_k} v^p{}_i + \frac{m}{M} v^e{}_k \frac{\partial}{\partial x_k} v^e{}_i \right) = n\, M \left[v^p{}_k \frac{\partial}{\partial x_k} v^p{}_i \left(\frac{m}{M}\right)^0 + O\left(\left(\frac{m}{M}\right)^1\right) \right]$$

1st-order term of $\frac{m}{M}$

$$(\rho^{MHD} v^{MHD}{}_k) \frac{\partial}{\partial x_k} v^{MHD}{}_i = n\,(M+m) \frac{M\, v^p{}_k + m\, v^e{}_k}{M+m} \frac{\partial}{\partial x_k} \left(\frac{M\, v^p{}_i + m\, v^e{}_i}{M+m} \right)$$

$$= n\, M \left(v^p{}_k + \frac{m}{M} v^e{}_k \right) \frac{\partial}{\partial x_k} \left(\frac{v^p{}_i + \frac{m}{M} v^e{}_i}{1+\frac{m}{M}} \right) = n\, M\, v^p{}_k \frac{\partial}{\partial x_k} v^p{}_i \left(\frac{m}{M}\right)^0 + O\left(\left(\frac{m}{M}\right)^1, \left(\frac{m}{M}\right)^2, \ldots \right)$$

1st-order and even higher-order terms of $\frac{m}{M}$

$$(\rho^p v^p{}_k) \frac{\partial}{\partial x_k} v^p{}_i + (\rho^e v^e{}_k) \frac{\partial}{\partial x_k} v^e{}_i \approx (\rho^{MHD} v^{MHD}{}_k) \frac{\partial}{\partial x_k} v^{MHD}{}_i$$

based on the 0th-order ($\frac{m}{M}$) approximation

$$\rho^{MHD} \left[\frac{\partial}{\partial t} \boldsymbol{v}^{MHD} + \left(\boldsymbol{v}^{MHD} \bullet \nabla \right) \boldsymbol{v}^{MHD} \right] = \boldsymbol{j}^{MHD} \times \boldsymbol{B} - \nabla P^{MHD}$$

Momentum eq. in MHD