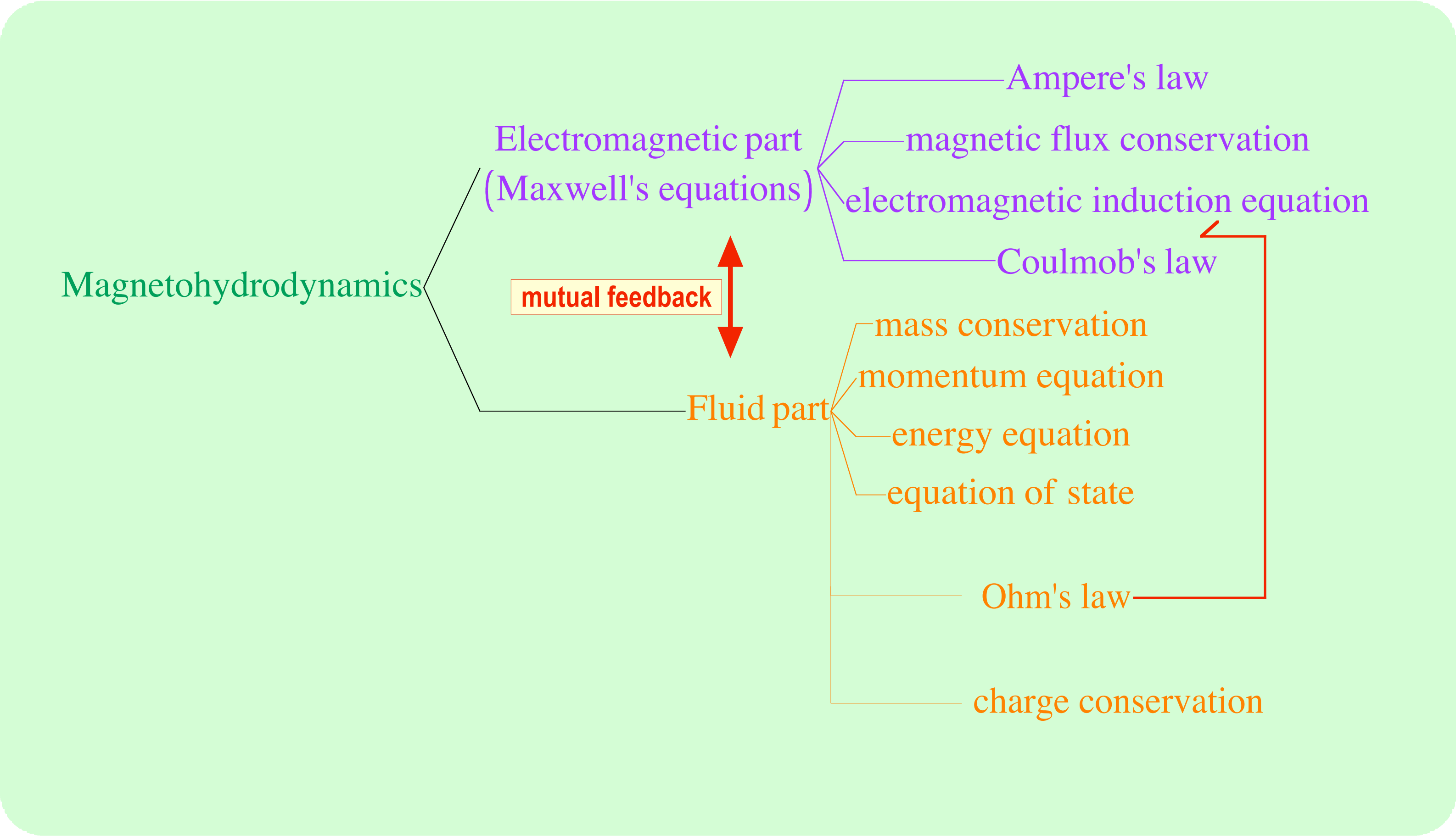
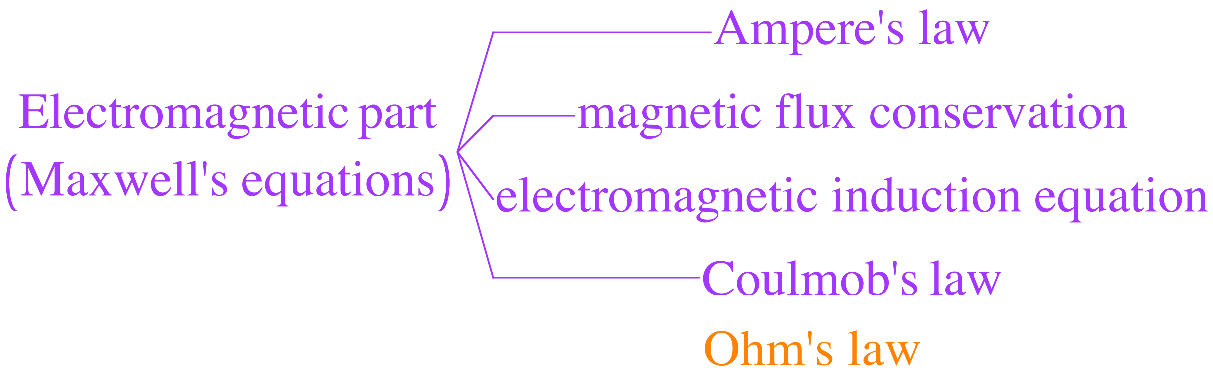
Magnetohydrodynamics

MHD equations

MHD equations...

Fluid (plasma) affects electromagnetic field, while electromagnetic field affects plasma.





(Plasma => Field)

Maxwell's equations (determine electromagnetic field's state from plasma's state)

MKS unit

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$ **... Generalized Ampere's law**

$$\nabla \bullet \boldsymbol{B} = 0$$ **... Conservation of magnetic flux**

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ **... Electromagnetic induction equation (Faraday's law)**

$$\nabla \bullet \boldsymbol{E} = \frac{\rho_c}{\varepsilon_0}$$ **... Coulomb's law**

$$\varepsilon_0 = 8.854\times 10^{-12}\ \mathrm{F\ m^{-1}}$$ **... permittivity of free space**

$$\mu_0 = 4\pi\times 10^{-7}\ \mathrm{H\ m^{-1}}$$ **... magnetic permeability**

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.998\times 10^{8}\ \mathrm{m\ s^{-1}}$$ **... speed of light**

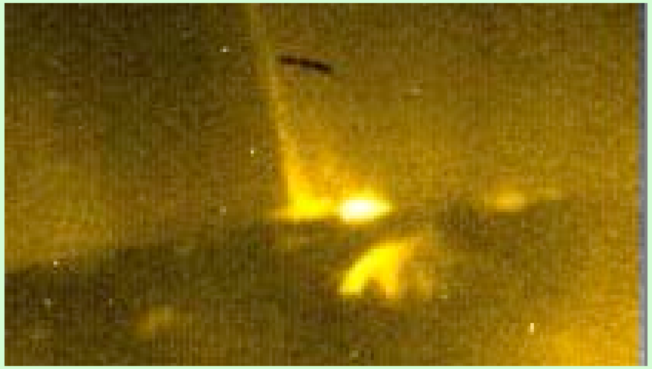
Unit:

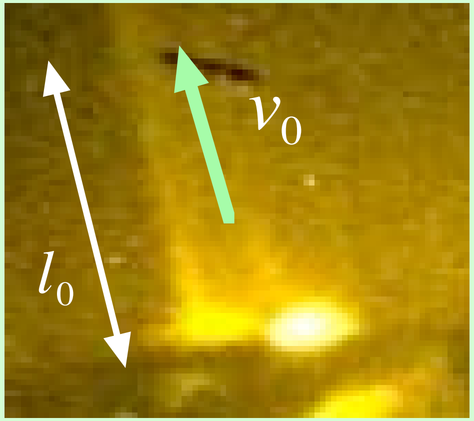
E... V/m, B... T (= 10^4 G), j... A/m^2

Typical scales in MHD

MHD phenomena... large-scale, slowly evolving phenomena

e.g. solar coronal jet





Typical scale:

length... $l_0 \sim$ 10^4 km

velocity... $v_0 \sim$ 100 km/s

time... $t_0 = \frac{l_0}{v_0} \sim$ 100 s

r_B (gyration radius of a proton) ~ 1 m

t_B (gyration period of a proton) ~ 10^{-4} s

in the solar corona

Approximations in MHD

- $v_0 << c$ **(non-relativistic approximation)**

Comparison between electromagnetic terms in the generalized Ampere's law

$$\boldsymbol{j} = \frac{1}{\mu_0}\left(\nabla\times\boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}\right)$$

Faraday's law

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \rightarrow \frac{E_0}{l_0} \sim \frac{B_0}{t_0}$$

$$\frac{B_0}{l_0} \qquad \frac{1}{c^2}\frac{E_0}{t_0} \longrightarrow \frac{1}{c^2}\frac{E_0}{t_0} \approx \frac{1}{c^2}\frac{1}{t_0}\frac{l_0}{t_0}B_0 = \frac{1}{c^2}\frac{l_0^2}{t_0^2}\frac{B_0}{l_0} = \frac{v_0^2}{c^2}\frac{B_0}{l_0} << \frac{B_0}{l_0}$$

$$\therefore \frac{B_0}{l_0} >> \frac{1}{c^2}\frac{E_0}{t_0} \Rightarrow we\ neglect\ \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

=> Propagation of electromagnetic waves (radiation) is <u>not described</u> by MHD.

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \Rightarrow \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$

Ampere's law in MHD

(derive magnetic field from current density)

• $\rho_c \sim 0$ (local charge neutrality)

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_c}{\varepsilon_0} \Rightarrow \frac{E_0}{l_0} \sim \frac{e\,(n_+ - n_-)}{\varepsilon_0}$$

$$n_+ - n_- \sim \frac{\varepsilon_0 E_0}{e\, l_0} \approx \frac{\varepsilon_0 \left(\frac{l_0 B}{t_0}\right)}{e\, l_0} = \frac{\varepsilon_0 v_0 B}{e\, l_0}$$

Condition of local charge neutrality:

$$n_+ - n_- << n_+ + n_- \equiv n_{total} \text{ (total number density)}$$

$$\therefore \frac{\varepsilon_0 v_0 B}{e\, l_0} << n_{total} \quad \rightarrow \quad 6.7\times10^{7} \frac{v_0 B}{l_0} \ll n_{total}$$

MKS unit

$e = 1.6\times10^{-19}$ C (electron charge)

e.g. solar corona: $n_{total} \sim 10^{14}$ m^{-3}, $v_0 \sim 10^{5}$ m/s, $l_0 \sim 10^{7}$ m, $B \sim 10^{-2}$ T

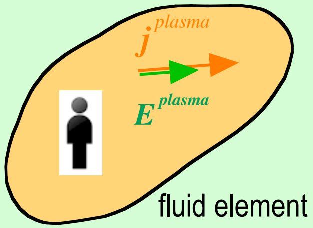
$$6.7\times10^{7} \frac{v_0 B}{l_0} << n_{total} \quad \rightarrow \quad 6.7\times10^{3} << 10^{14} \ldots$$ *satisfied*

Coulomb's law in MHD $\nabla \cdot \boldsymbol{E} = \frac{\rho_c}{\varepsilon_0} \sim 0$

(does NOT derive electric field from charge density => Ohm's law)

Ohm's law

Plasma frame



$$E^{plasma} = \eta\, j^{plasma} = \frac{1}{\sigma} j^{plasma}$$

η... *resistivity* $(\mathrm{ohm\ m = kg\ m^3\ s^{-3}\ A^{-2}})$

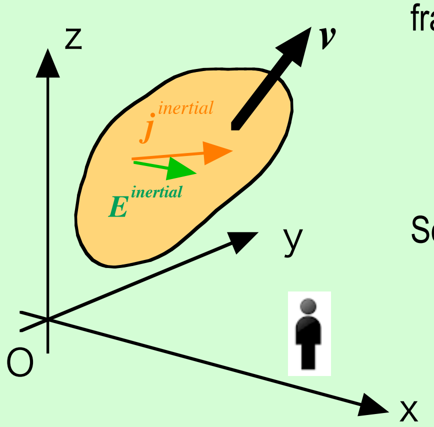
σ... *electric conductivity* $(\mathrm{ohm^{-1}\ m^{-1} = mho\ m^{-1}})$

=> plasma's property determined by kinetic approach

Electric field is proportional to electric current density.

Ohm's law in plasma frame

Inertial frame



When the fluid element moves at the **constant velocity v** against inertial frame, then

$$j^{inertial} = j^{plasma}$$

non-relativistic, charge-neutral case

$$E^{inertial} = E^{plasma} - v \times B^{plasma} \qquad B^{inertial} = B^{plasma}$$

So,

$$E^{inertial} = \eta\, j^{inertial} - v \times B^{inertial}$$

Ohm's law in inertial frame

$-v \times B^{inertial}$ ... plays a role in transporting magnetic field at $v_{\perp}$

(derive electric field from flow velocity and current density)

MHD induction equation

... represents the electromagnetic part of MHD

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ Faraday's law

$$\boldsymbol{E}^{MHD} = \eta\, \boldsymbol{j}^{MHD} - \boldsymbol{v} \times \boldsymbol{B}$$ Ohm's law

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \left(-\boldsymbol{v} \times \boldsymbol{B} + \eta\, \boldsymbol{j}^{MHD}\right)$$

$$\nabla \times \boldsymbol{B} = \mu_0\, \boldsymbol{j}^{MHD}$$ Ampere's law

eliminate $\boldsymbol{j}^{MHD}$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times \left(\boldsymbol{v} \times \boldsymbol{B} - \eta_{diff} \nabla \times \boldsymbol{B}\right)$$

$\eta_{diff} \equiv \frac{\eta}{\mu_0}$ *... magnetic diffusivity*

MHD induction equation

... describes evolution of magnetic field using plasma's state