Thermal conduction... typical diffusion process

Temperature flux: $\boldsymbol{F}_T = -\alpha \nabla T$

... from a high temperature region to low temperature region

α: **thermal diffusivity**... dimension: $\boldsymbol{v} \times l$ (velocity x mean free path)

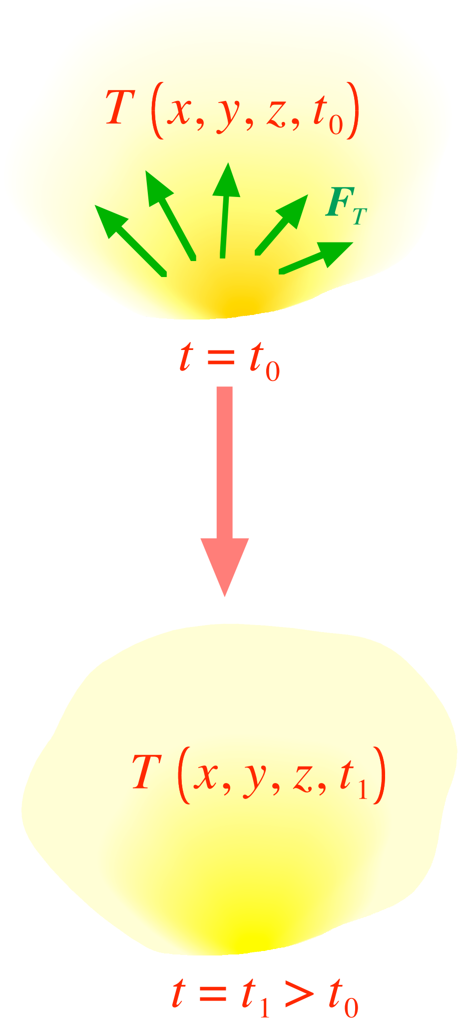
Temperature change rate: $\dfrac{\partial T}{\partial t}$

= net influx (influx – outflux): $-\nabla \bullet \boldsymbol{F}_T$

$$\nabla \bullet \boldsymbol{F}_T \equiv \frac{\partial F_{Tx}}{\partial x} + \frac{\partial F_{Ty}}{\partial y} + \frac{\partial F_{Tz}}{\partial z}$$ ... divergence of $\boldsymbol{F}_T$

$$\frac{\partial T(x, y, z, t)}{\partial t} = \alpha \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) T(x, y, z, t)$$

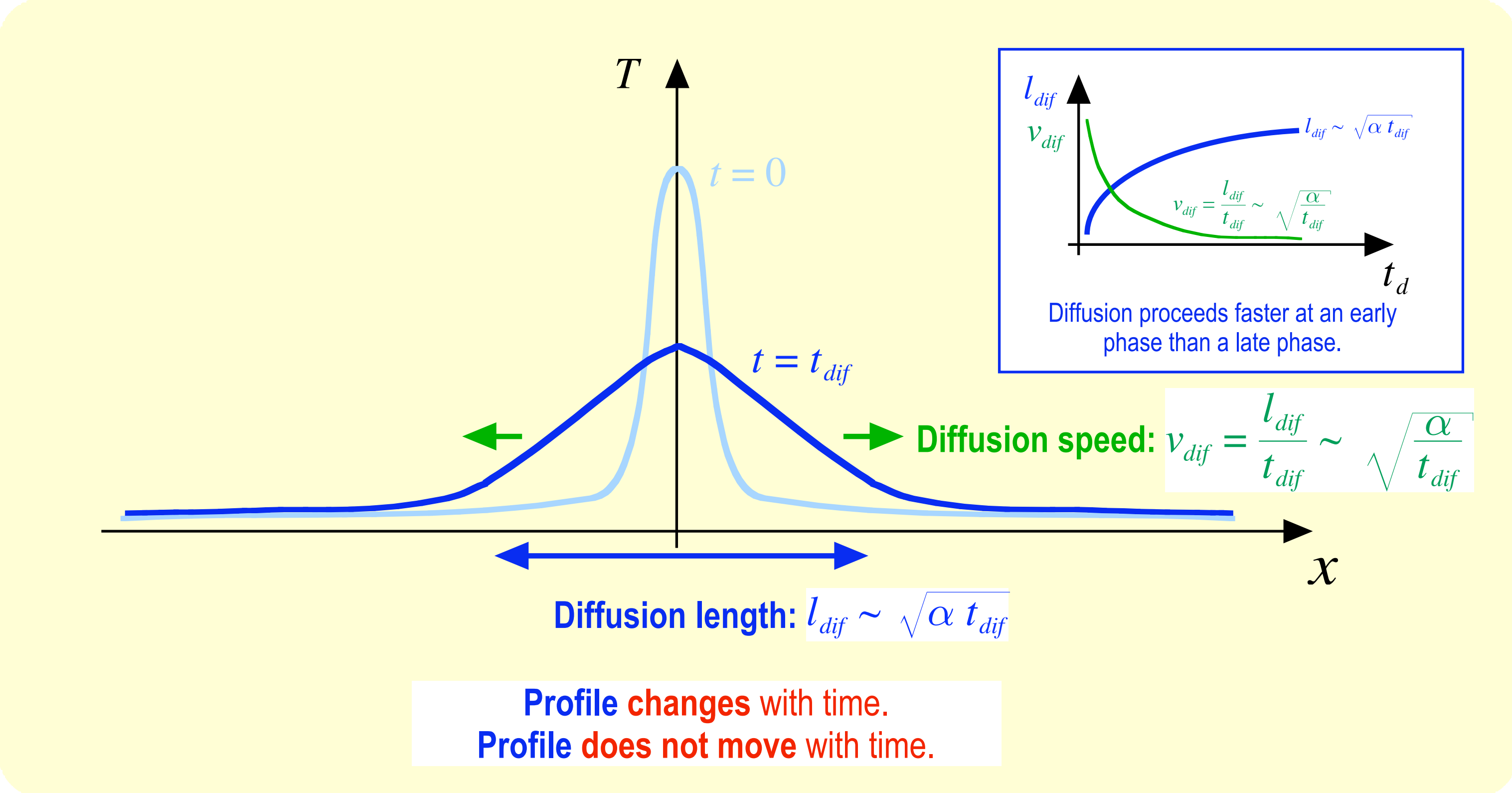
Time-dependent part... 1st-order partial derivative
Space-dependent part... 2nd-order partial derivative



Evolutionary profile of a diffusion process

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \Rightarrow \frac{\Delta T}{t_{dif}} \sim \alpha \frac{\Delta T}{l_{dif}^2}$$

order estimate
$\left(\frac{\partial}{\partial t} \sim \frac{1}{t_{dif}}, \frac{\partial}{\partial x} \sim \frac{1}{l_{dif}}\right)$



Wave equation

Wave propagation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Time-dependent part... 2nd-order partial derivative
Space-dependent part... 2nd-order partial derivative

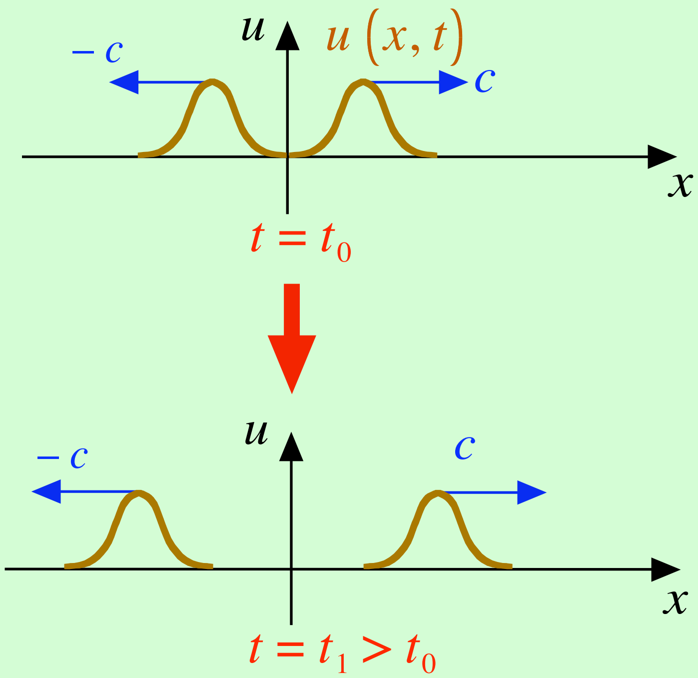
In a three-dimensional case,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0$$

Evolutionary profile of a propagating wave

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,\ \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0$$

u: physical quantity
c: propagation speed



Profile does not change with time.
Profile moves with time.

Linear PDE and Nonlinear PDE

Linear PDE... linear for solution function and its partial derivatives

Example: **2nd-order linear PDE** (solution function: $u = u(x, y)$)

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)\,u = G(x,y)$$

$G(x,y) = 0 \Rightarrow homogeneous$ → principle of superposition

$G(x,y) \neq 0 \Rightarrow inhomogeneous$

quasi-linear PDE... linear for the highest-order partial derivative of solution function

$$A, B, C, D, E, F, G \ldots \text{ depends on } \left(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u\right)$$

e.g. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ This is the 1st-order PDE, so the highest-order is the 1st order.

semi-linear PDE... coefficients of the highest-order partial derivative do not depend on solution function

$$A, B, C \ldots \text{ depends on } (x, y)$$

e.g. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0$ (*Burgers equation*) This is the 2nd-order PDE, so the highest-order is the 2nd order.

ν: *constant*

Nonlinear PDE... any PDE except for linear, quasi-linear, semi-linear PDEs

Example: **2nd-order nonlinear PDE** (solution function: $u = u(x, y)$)

e.g., $\left[\frac{\partial u}{\partial x}\right]^2 + \left[\frac{\partial u}{\partial y}\right]^2 = u$

$\left[\frac{\partial u}{\partial x}\right]^2 + \left[\frac{\partial u}{\partial y}\right]^2 = 1$ (eikonal equation)

This is the 1st-order PDE, so the highest-order is the 1st order.