Partial differential equations

some basics

0. Typical PDEs used in physics

1. Basic features

- Three types of PDEs (hyperbolic, parabolic, elliptic)
- Initial condition, boundary condition
- Characteristics

2. Hyperbolic PDE

3. Parabolic PDE

4. Elliptic PDE

5. Traditional methods for solving PDEs

Typical PDEs used in physics

Poisson equation & Laplace equation

Static electric field

$\rho_c(x, y, z)$... charge density

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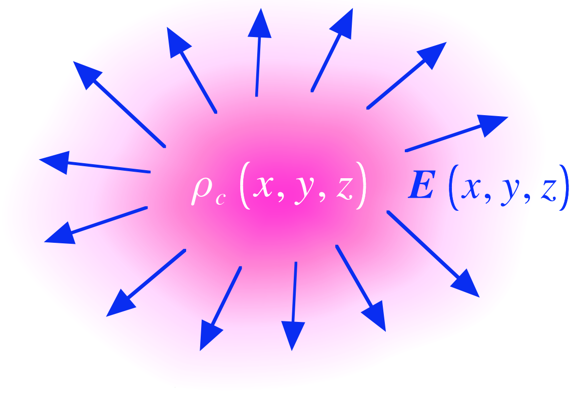
$\boldsymbol{E}(x, y, z)$... static electric field



$\nabla^2 \equiv \Delta$: *Laplacian*

$$\boldsymbol{E} = -\nabla\phi_c, \quad \nabla \bullet \boldsymbol{E} = \frac{\rho_c}{\varepsilon_0} \longrightarrow \nabla^2\phi_c = \Delta\phi_c = -\frac{\rho_c}{\varepsilon_0}$$

ε_0: permittivity

$\phi_c(x, y, z)$: electrostatic potential

Poisson equation

Time-dependent part... does not exist (=> steady or static)

Space-dependent part... 2nd-order partial derivative

In a vacuum case,

$$\nabla^2\phi_c = \Delta\phi_c = 0$$

Laplace equation

solution is called **harmonic function**

Self-gravity field

mass density: $\rho\,(x, y, z)$

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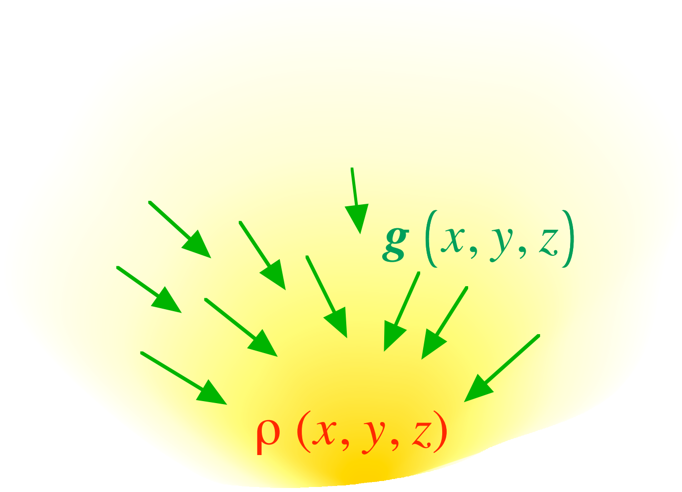
self-gravity field: $\boldsymbol{g}\,(x, y, z)$



$$\boldsymbol{g} = -\nabla\Phi, \quad \nabla \cdot \boldsymbol{g} = -4\pi G\rho \quad \longrightarrow \quad \nabla^2\Phi = 4\pi G\rho$$

Poisson equation

G: universal gravitational constant

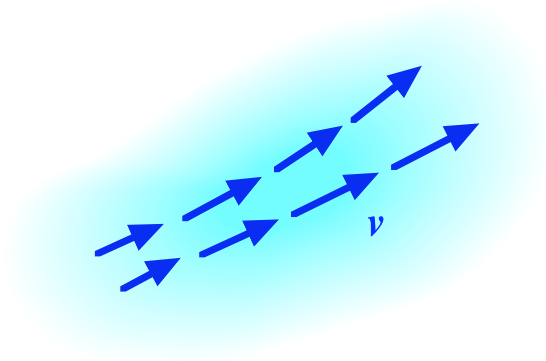
$\Phi\,(x, y, z)$: gravitational potential

Irrotational and incompressible flow

Irrotational flow (potential flow)... $\nabla \times \boldsymbol{v} \equiv \mathrm{rot}\ \boldsymbol{v} = \mathbf{0}$

$$\nabla \times \boldsymbol{v} \equiv \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \dfrac{\partial v_z}{\partial y} - \dfrac{\partial v_y}{\partial z} \\ \dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x} \\ \dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y} \end{pmatrix}$$

$$\boldsymbol{v} = -\nabla\Phi_v,\ \ \Phi_v\text{: } \textit{velocity potential} \ \Rightarrow\ \nabla \times \boldsymbol{v} = \mathbf{0}$$



Incompressible flow... $\nabla \bullet \boldsymbol{v} \equiv \mathrm{div}\ \boldsymbol{v} = 0$

$$\boldsymbol{v} = -\nabla\Phi_v$$

$$\nabla \bullet \boldsymbol{v} = 0$$

$$\Rightarrow\ \nabla \bullet \left(-\nabla\Phi_v\right) = -\nabla^2\Phi_v = 0$$

Laplace equation

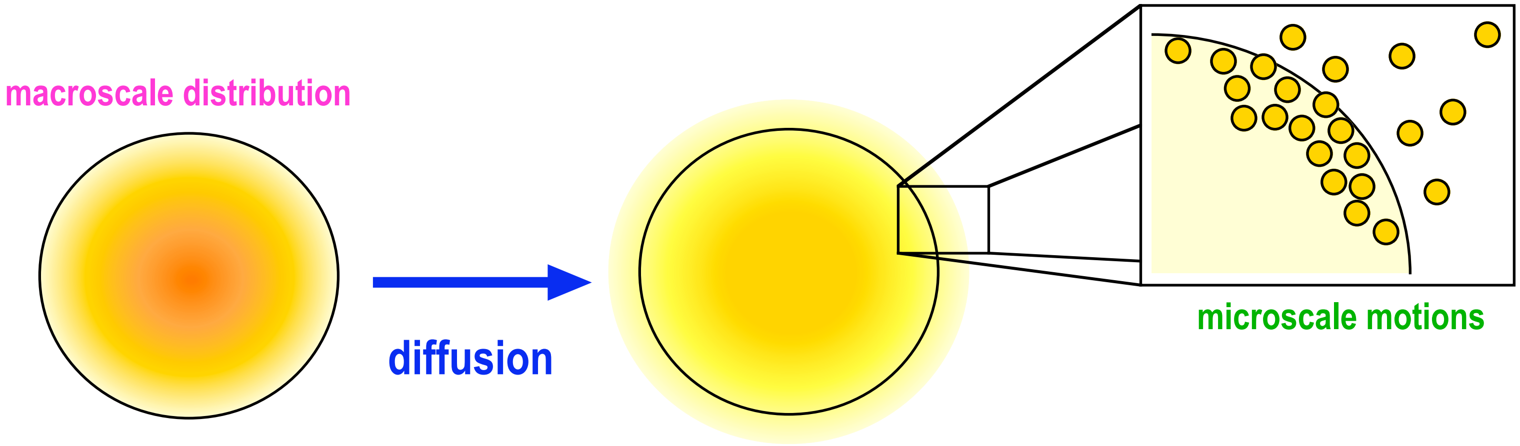
Diffusion equation

What is diffusion?

A time-dependent process

caused by **microscale motions of particles**

smoothes **macroscale distribution of particles**



Diffusion is characterized by

length l × *velocity* $\boldsymbol{v}$

l... mean free path

v... speed of a particle