Electric field in MHD => two major components

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}^{MHD} = \underbrace{-\nabla \times \left(-\boldsymbol{v} \times \boldsymbol{B}\right)}_{\text{convective term}} \underbrace{-\nabla \times \left(\eta_{diff} \nabla \times \boldsymbol{B}\right)}_{\text{diffusion term}}$$

convective term

$$\boldsymbol{E}_{conv} \equiv -\boldsymbol{v} \times \boldsymbol{B}$$

diffusion term

$$\boldsymbol{E}_{resis} \equiv \eta_{diff} \nabla \times \boldsymbol{B}$$

- **Convective term:** $-\nabla \times \boldsymbol{E}_{conv} = \nabla \times \left(\boldsymbol{v} \times \boldsymbol{B}\right)$

  it drives evolution of magnetic field via **macroscale motion (convection) of fluid elements.**

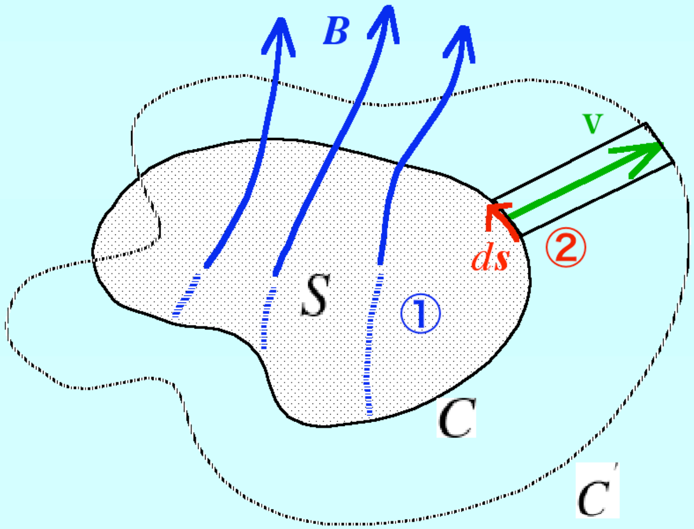
- **Diffusion term:** $-\nabla \times \boldsymbol{E}_{resis} = -\nabla \times \left(\eta_{diff} \nabla \times \boldsymbol{B}\right)$

  it drives evolution of magnetic field via **microscale motion (collision) of particles.**

Evolution of magnetic field via $\boldsymbol{E}_{conv} \equiv -\boldsymbol{v} \times \boldsymbol{B}$:

$$\Phi_S = \iint_S \boldsymbol{B} \bullet d\boldsymbol{S}$$

... total magnetic flux crossing S



Lagrangian derivative

$$\begin{cases} \text{Change of } B \text{ when } C \text{ is fixed ①} \\ + \\ \text{Change of } C \text{ when } B \text{ is fixed ②} \end{cases}$$

$$\begin{aligned}
\frac{d\Phi_S}{dt} &= \iint_S \frac{\partial \boldsymbol{B}}{\partial t} \bullet d\boldsymbol{S} + \oint_C \boldsymbol{B} \bullet (\boldsymbol{v} \times d\boldsymbol{s}) \quad (①, ②) \\
&= \iint_S \frac{\partial \boldsymbol{B}}{\partial t} \bullet d\boldsymbol{S} + \oint_C (\boldsymbol{B} \times \boldsymbol{v}) \bullet d\boldsymbol{s} \quad \text{(scalar triple vector product)} \\
&= \iint_S \frac{\partial \boldsymbol{B}}{\partial t} \bullet d\boldsymbol{S} + \iint_S \nabla \times (\boldsymbol{B} \times \boldsymbol{v}) \bullet d\boldsymbol{S} \quad \text{(Stokes's theorem)} \\
&= \iint_S \left[\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{B} \times \boldsymbol{v}) \right] \bullet d\boldsymbol{S} \\
&= \iint_S \left[\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \right] \bullet d\boldsymbol{S} \\
&= 0
\end{aligned}$$

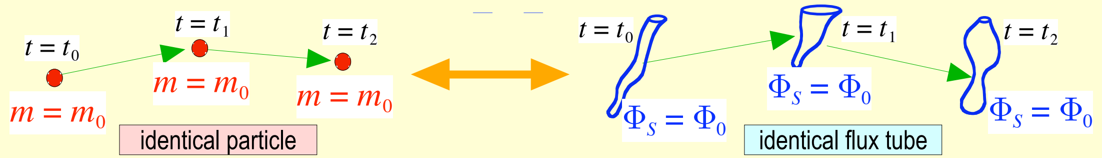
$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \Rightarrow \frac{d\Phi_S}{dt} = 0$$

Physical meaning of $\frac{d\Phi_S}{dt} = 0$ => Frozen-in evolution

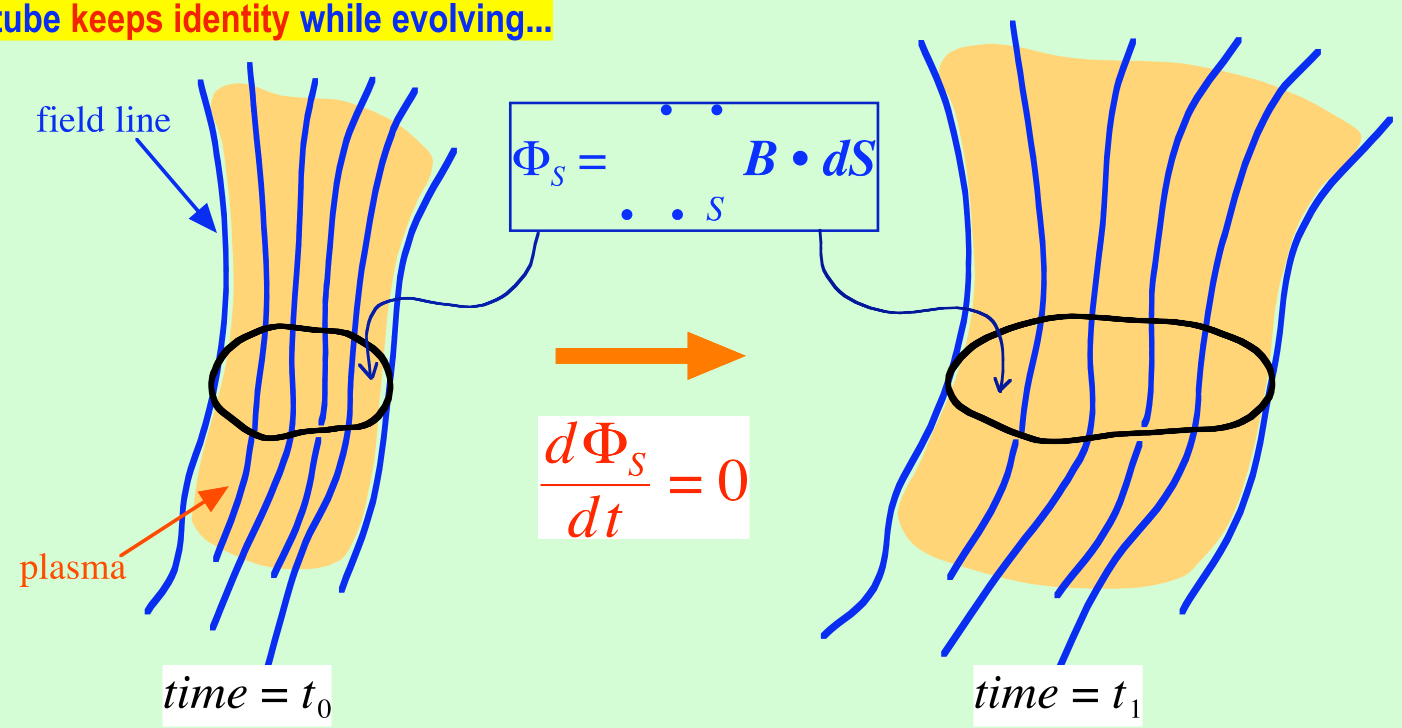
The **number of magnetic field lines** passing through the area bounded by **any closed curve moving together with a plasma does not change with time**.

⇓

We can determine an **identical flux tube** in which magnetic field lines **are frozen into** the plasma.



Flux tube keeps identity while evolving...



Evolution of magnetic field via $\boldsymbol{E}_{resis} \equiv \eta_{diff} \nabla \times \boldsymbol{B}$:

When η_{diff} is constant, $\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times (\eta_{diff} \nabla \times \boldsymbol{B}) \Rightarrow \frac{\partial \boldsymbol{B}}{\partial t} = \eta_{diff} \nabla^2 \boldsymbol{B}$

$\frac{\partial q}{\partial t} = \eta_{diff} \nabla^2 q$ **... diffusion equation**

$q = T, j_x, B_y, \ldots$

Time scale of diffusion: τ_{diff}

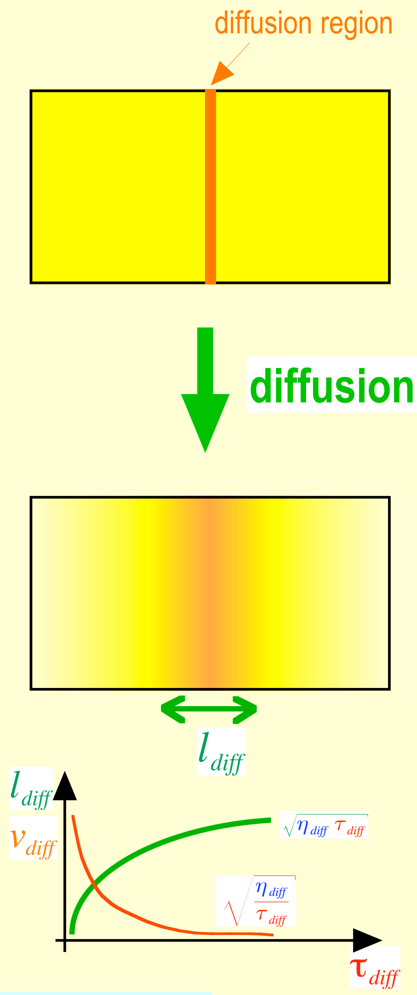
$$\frac{\partial q}{\partial t} = \eta_{diff} \nabla^2 q \Rightarrow \frac{q}{\tau_{diff}} \sim \eta_{diff} \frac{q}{l_{diff}^2} \rightarrow \tau_{diff} \sim \frac{l_{diff}^2}{\eta_{diff}}$$

l_{diff}: *width of diffusion region*

Length scale of diffusion: $l_{diff} \sim \sqrt{\eta_{diff} \tau_{diff}}$

Velocity scale of diffusion: v_{diff}

$$v_{diff} \sim \frac{l_{diff}}{\tau_{diff}} = \frac{\sqrt{\eta_{diff} \tau_{diff}}}{\tau_{diff}} = \sqrt{\frac{\eta_{diff}}{\tau_{diff}}}$$



Diffusion proceeds fast at the beginning and then slowly.

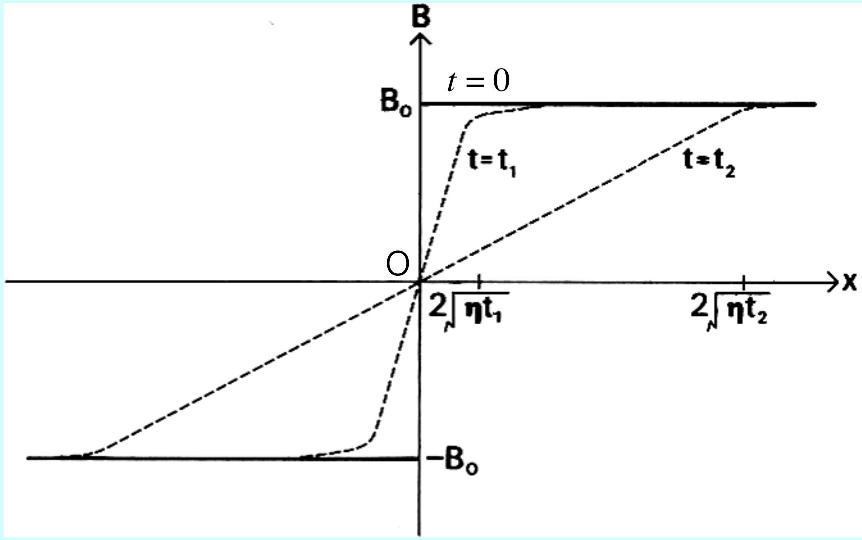
Diffusion of an antiparallel magnetic field (annihilation)

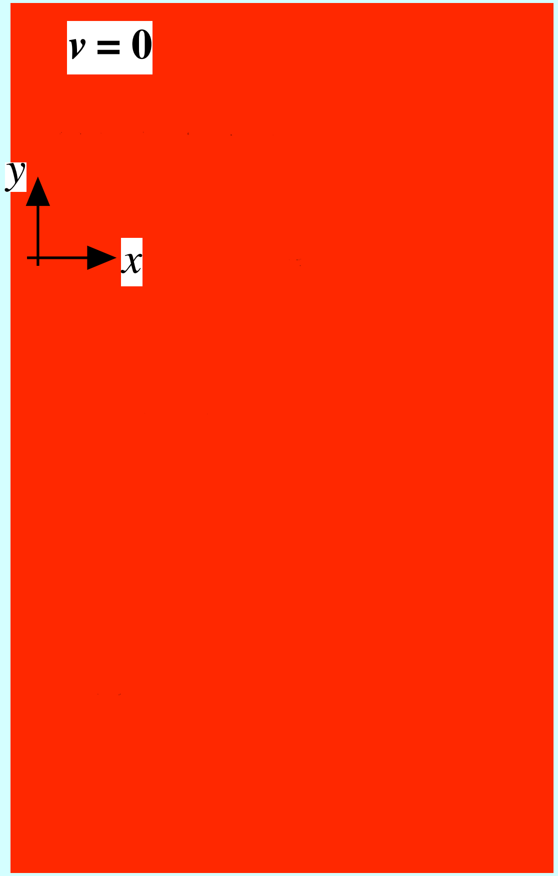
$$\frac{\partial B_y}{\partial t} = \eta_{diff} \frac{\partial^2}{\partial x^2} B_y$$

$B_y(x, t=0) = \begin{cases} B_0 & for\ x>0 \\ -B_0 & for\ x<0 \end{cases}$ ... initial condition

$B_y(x=\pm\infty, t) = \begin{cases} B_0 & for\ x=\infty \\ -B_0 & for\ x=-\infty \end{cases}$ ... boundary condition

$$B_y(x,t) = \frac{2B_0}{\sqrt{\pi}} \operatorname{erf}\left(\frac{x}{\sqrt{4\eta_{diff}\, t}}\right), \quad \operatorname{erf}(\xi) = \int_0^{\xi} e^{-u^2} du$$





Solar MHD (Priest 1982)

- magnetic field is annihilated at x = 0
- magnetic field diffuses through a plasma => **violates frozen-in evolution**