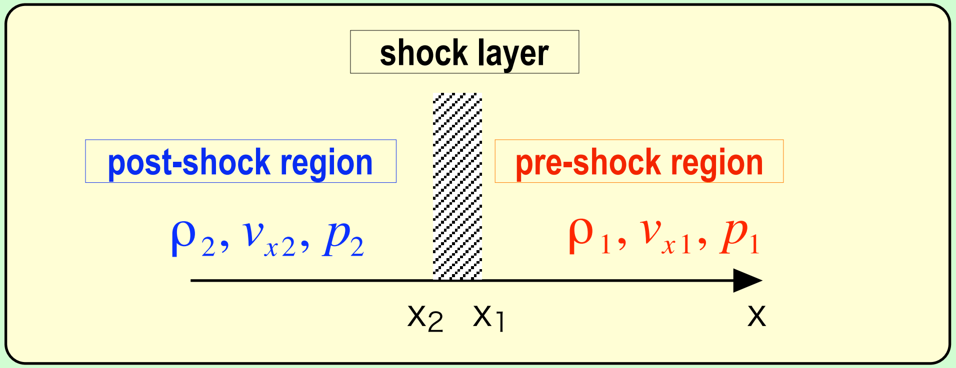
Relations between pre-shock and post-shock regions

=> **Rankine-Hugoniot relations**

$$\rho_1 v_{x1} = \rho_2 v_{x2}$$

$$\rho_1 v_{x1}^2 + p_1 = \rho_2 v_{x2}^2 + p_2$$

$$\left(\frac{1}{2}\rho_1 v_{x1}^2 + \frac{\gamma}{\gamma - 1} p_1\right) v_{x1} = \left(\frac{1}{2}\rho_2 v_{x2}^2 + \frac{\gamma}{\gamma - 1} p_2\right) v_{x2}$$



From these three equations, we derive three ratios of physical quantities $\frac{\rho_2}{\rho_1}, \frac{v_{x2}}{v_{x1}}, \frac{p_2}{p_1}$ across the shock layer.

From R-H relations, we obtained the following equation for $X = \frac{\rho_2}{\rho_1}$

$$\left(\frac{\gamma - 1}{2} M_1^2 + 1\right) X^2 - \left(\gamma M_1^2 + 1\right) X + \frac{\gamma + 1}{2} M_1^2 = 0$$

$M_1 \equiv \frac{v_{x1}}{\sqrt{\gamma\, p_1/\rho_1}} = \frac{v_{x1}}{c_{s1}}$ is Mach number in the pre-shock region.

There are two mathematical solutions, namely,

$$X = \frac{\left(\gamma + 1\right) M_1^2}{\left(\gamma - 1\right) M_1^2 + 2} \quad or \quad X = 1$$

The former gives a non-trivial solution of shock process.

$$\frac{\rho_2}{\rho_1} = \frac{\left(\gamma + 1\right) M_1^2}{2 + \left(\gamma - 1\right) M_1^2}$$

$$\frac{v_{x2}}{v_{x1}} = \frac{2 + \left(\gamma - 1\right) M_1^2}{\left(\gamma + 1\right) M_1^2}$$

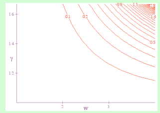
$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 - \left(\gamma - 1\right)}{\gamma + 1}$$

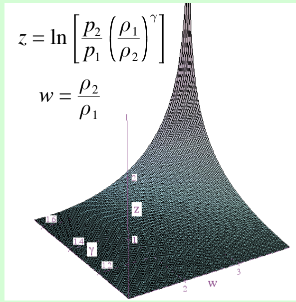
Properties of HD shock wave

Entropy ($s = C_V \ln \frac{p}{\rho^\gamma} + const$) ***increases*** *from* ***pre-shock region*** *to* ***post-shock region***

=> **irreversible process** (**thermalization**: ordered flow energy => random motion energy)

$$s_2 - s_1 \propto \ln\left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right] \geq 0 \text{ when } \frac{\rho_2}{\rho_1} \geq 1; \quad \frac{p_2}{p_1} = \frac{1 - \frac{\gamma+1}{\gamma-1}\frac{\rho_2}{\rho_1}}{-\frac{\gamma+1}{\gamma-1} + \frac{\rho_2}{\rho_1}} \geq 1$$





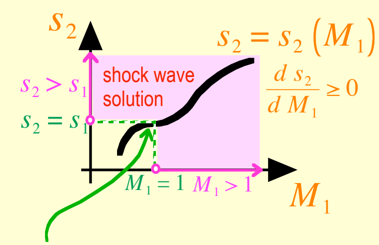
$\frac{s_2}{s_1} \geq 1$ is consistent with the *2nd law of thermodynamics*.

$M_1 > 1$ *in the* ***pre-shock region***, $M_2 < 1$ *in the* ***post-shock region***

$$\frac{ds_2}{dM_1} = \frac{C_V}{p_2}\frac{dp_2}{dM_1} - \gamma\frac{C_V}{\rho_2}\frac{d\rho_2}{dM_1} = \frac{4\,C_V\,\gamma\,(\gamma-1)\left(M_1^2-1\right)^2}{M_1\left[2\gamma M_1^2 - (\gamma-1)\right]\left[2+(\gamma-1)\,M_1^2\right]} \geq 0$$

> 0 *because* $p_2 > 0$

$s_2(M_1)$ is a monotonically increasing function of M_1.



When $M_1 = 1$ (trivial solution),

$\rho_2 = \rho_1 \quad v_{x2} = v_{x1}$

$p_2 = p_1 \quad s_2 = s_1$

$$\frac{ds_2}{dM_1} = 0$$

$$\therefore \frac{s_2}{s_1} > 1 \Rightarrow M_1 > 1$$

$$M_2^2 \equiv \frac{v_{x2}^2}{c_{s2}^2} = 1 - \frac{(\gamma+1)\left(M_1^2-1\right)}{2\gamma M_1^2 - (\gamma-1)} < 1$$

> 0 *because* $p_2 > 0$

pre-shock region... supersonic

post-shock region... subsonic

***Density ratio* has *an upper limit*:**

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)\, M_1^2}{2 + (\gamma - 1)\, M_1^2}$$

... **upper limit** $\frac{\gamma + 1}{\gamma - 1}$ when $M_1 \rightarrow \infty$

$$\frac{v_{x2}}{v_{x1}} = \frac{2 + (\gamma - 1)\, M_1^2}{(\gamma + 1)\, M_1^2}$$

... **lower limit** $\frac{\gamma - 1}{\gamma + 1}$ when $M_1 \rightarrow \infty$

***Pressure (and temperature) ratio* has *no upper limit*:**

$$\frac{p_2}{p_1} = \frac{2\,\gamma\, M_1^2 - (\gamma - 1)}{\gamma + 1}$$

... **no upper limit** when $M_1 \rightarrow \infty$