4) Energy equation

proton's energy conservation eq.

$$\frac{\partial}{\partial t}\left(\frac{n\,M}{2}\,u^{p2}+n\,M\,\varepsilon^{p}\right)+\frac{\partial}{\partial x_k}\left[\left(\frac{n\,M}{2}\,u^{p2}+n\,M\,\varepsilon^{p}+P^{p}\right)u^{p}{}_{k}-\pi^{p}_{ik}\,u^{p}_{i}+\mathscr{F}^{p}_{k}\right]-n\,M\left\langle\frac{F^{p}_{k}}{M}\,v^{p}_{k}\right\rangle=\int_v\frac{1}{2}\,M\,v^{p2}\left(\frac{\delta f^{p}}{\delta t}\right)_c d^3v^{p}$$

electron's energy conservation eq.

$$\frac{\partial}{\partial t}\left(\frac{n\,m}{2}\,u^{e2}+n\,m\,\varepsilon^{e}\right)+\frac{\partial}{\partial x_k}\left[\left(\frac{n\,m}{2}\,u^{e2}+n\,m\,\varepsilon^{e}+P^{e}\right)u^{e}{}_{k}-\pi^{e}_{ik}\,u^{e}_{i}+\mathscr{F}^{e}_{k}\right]-n\,m\left\langle\frac{F^{e}_{k}}{m}\,v^{e}_{k}\right\rangle=\int_v\frac{1}{2}\,m\,v^{e2}\left(\frac{\delta f^{e}}{\delta t}\right)_c d^3v^{e}$$

Add both equations (under the 0-the order approximation of $\frac{m}{M} << 1$)...

$$\left(\frac{n\,M}{2}\,u^{p2}+n\,M\,\varepsilon^{p}\right)+\left(\frac{n\,m}{2}\,u^{e2}+n\,m\,\varepsilon^{e}\right)\approx\frac{1}{2}\,\rho^{MHD}\,u^{MHD2}+\rho^{MHD}\,\varepsilon^{MHD}$$

$$\left(\frac{n\,M}{2}\,u^{p2}\,u^{p}{}_{k}\right)+\left(\frac{n\,m}{2}\,u^{e2}\,u^{e}{}_{k}\right)\approx\frac{1}{2}\,\rho^{MHD}\,u^{MHD2}\,u^{MHD}{}_{k}$$

assumption: $T^{p}\sim T^{e}\sim T^{MHD}$

$$\left[\left(n\,M\,\varepsilon^{p}+P^{p}\right)u^{p}{}_{k}\right]+\left[\left(n\,m\,\varepsilon^{p}e+P^{e}\right)u^{e}{}_{k}\right]=\frac{\gamma}{\gamma-1}\,P^{p}u^{p}{}_{k}+\frac{\gamma}{\gamma-1}\,P^{e}u^{e}{}_{k}=\frac{\gamma}{\gamma-1}\,\frac{P^{MHD}}{2}\left[2\,u^{p}{}_{k}-\left(u^{p}{}_{k}-u^{e}{}_{k}\right)\right]=\frac{\gamma}{\gamma-1}\,P^{MHD}\left(u^{p}_{k}-\frac{u^{p}{}_{k}-u^{e}{}_{k}}{2}\right)$$

$$\approx\left(\rho^{MHD}\,\varepsilon^{MHD}+P^{MHD}\right)u^{MHD}{}_{k}$$

assumption:

$$\left|\frac{\partial}{\partial x_k}\left(P^{MHD}\,u^{p}_{k}\right)\right| \gg \left|\frac{k_B}{e}\,j_k\,\frac{\partial T}{\partial x_k}\right|$$

$$\left(\pi^{p}_{ik}\,u^{p}_{i}\right)+\left(\pi^{e}_{ik}\,u^{e}_{i}\right)\approx\pi^{p}_{ik}\,u^{p}_{i}\approx\pi^{MHD}_{ik}\,u^{MHD}_{i}$$

assumption: $D^{p}_{ik}\sim D^{MHD}_{ik}$, $\mu^{MHD}\sim\mu^{p}\gg\mu^{e}$ $\left(\sqrt{M}\gg\sqrt{m}\right)$

$$\left(\mathscr{F}^{p}_{k}\right)+\left(\mathscr{F}^{e}_{k}\right)\approx\mathscr{F}^{e}_{k}\approx\mathscr{F}^{MHD}_{k}$$

assumption: $T^{p}\sim T^{e}\sim T^{MHD}$ and $\kappa^{MHD}\sim\kappa^{e}\gg\kappa^{p}$ $\left(1/\sqrt{m}\gg 1/\sqrt{M}\right)$

$$\left(n\,M\left\langle\frac{F^{p}_{k}}{M}\,v^{p}_{k}\right\rangle\right)+\left(n\,m\left\langle\frac{F^{e}_{k}}{m}\,v^{e}_{k}\right\rangle\right)=n\,e\,E_k\left(u^{p}_{k}-u^{e}_{k}\right)+n\,g_k\left(M\,u^{p}_{k}+m\,u^{e}_{k}\right)\approx E_k\,j^{MHD}_{k}+\rho^{MHD}\,g_k\,u^{MHD}_{k}$$

$$\int_v\frac{1}{2}\,M\,v^{p2}\left(\frac{\delta f^{p}}{\delta t}\right)_c d^3v^{p}+\int_v\frac{1}{2}\,m\,v^{e2}\left(\frac{\delta f^{e}}{\delta t}\right)_c d^3v^{e}=0$$

Total energy is conserved via collisions between protons and electrons.

$$\frac{\partial}{\partial t}\left(\frac{\rho^{MHD}}{2}\,u^{MHD2}+\rho^{MHD}\,\varepsilon^{MHD}\right)+\frac{\partial}{\partial x_k}\left[\left(\frac{\rho^{MHD}}{2}\,u^{MHD2}+\rho^{MHD}\,\varepsilon^{MHD}+P^{MHD}\right)u^{MHD}{}_{k}-\pi^{MHD}_{ik}\,u^{MHD}_{i}+\mathscr{F}^{MHD}_{k}\right]=\boldsymbol{E}^{MHD}\cdot\boldsymbol{j}^{MHD}+\rho^{MHD}\,\boldsymbol{g}\cdot\boldsymbol{u}^{MHD}$$

Plasma energy equation

E • j... represents electromagnetic contribution to the energy equation:

Electromagnetic energy equation: $$\frac{\partial}{\partial t}\left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right) + \nabla \bullet \underbrace{\left(\boldsymbol{E} \times \frac{\boldsymbol{B}}{\mu_0}\right)}_{\text{Poynting flux}} = \underbrace{-\,\boldsymbol{E} \bullet \boldsymbol{j}}_{\text{source term}}$$

source term
(plasma => electromagnetic field)

$$\underbrace{\boldsymbol{E}^{MHD} \bullet \boldsymbol{j}^{MHD}}_{\text{source term}} = -\left(\boldsymbol{u}^{MHD} \times \boldsymbol{B}\right) \bullet \boldsymbol{j}^{MHD} + \eta\, \boldsymbol{j}^{MHD} \bullet \boldsymbol{j}^{MHD} + \underbrace{\frac{1}{e\,n}\left[\boldsymbol{j}^{MHD} \times \boldsymbol{B}\right] \bullet \boldsymbol{j}^{MHD}}_{=0}$$

source term
(electromagnetic field => plasma)

$$\boldsymbol{E}^{MHD} \bullet \boldsymbol{j}^{MHD} = \eta\, j^{MHD\,2} + \left(\boldsymbol{j}^{MHD} \times \boldsymbol{B}\right) \bullet \boldsymbol{u}^{MHD}$$

$\eta\, j^{MHD\,2}$... **Ohmic heating rate** which reduces electromagnetic energy and increases internal energy of plasma => **source term** in **internal energy equation**

$\left(\boldsymbol{j}^{MHD} \times \boldsymbol{B}\right) \bullet \boldsymbol{u}^{MHD}$... **work rate done by jxB force**, which reduces electromagnetic energy and increases flow energy of plasma => **source term** in **flow energy equation**

$$\frac{\partial}{\partial t}\left(\rho^{MHD}\, \varepsilon^{MHD}\right) + \frac{\partial}{\partial x_k}\, \rho^{MHD}\, \varepsilon^{MHD}\, u^{MHD}_{k} = -P^{MHD}\frac{\partial u^{MHD}_{k}}{\partial x_k} - \frac{\partial \mathscr{F}^{MHD}_{k}}{\partial x_k} + \Psi^{MHD} + \eta\, j_{MHD}^2$$

Internal energy eq. in MHD

5) Charge conservation equation

$$\frac{e}{M} \times \quad \frac{\partial \rho^p}{\partial t} + \nabla \bullet \left(\rho^p \boldsymbol{u}^p\right) = 0$$ ... proton's mass conservation $\rho^p \equiv n_p M$

$$\frac{e}{m} \times \quad \frac{\partial \rho^e}{\partial t} + \nabla \bullet \left(\rho^e \boldsymbol{u}^e\right) = 0$$ ... electron's mass conservation $\rho^e \equiv n_e m$

subtraction

$$\frac{\partial \rho_c}{\partial t} + \nabla \bullet \boldsymbol{j} = 0$$

$$\rho_c \equiv e\left(n_p - n_e\right)$$

$$\boldsymbol{j} \equiv e\left(n_p \boldsymbol{u}^p - n_e \boldsymbol{u}^e\right)$$

Charge conservation equation

In MHD, $n_p \sim n_e \sim n$, so $\rho_c^{MHD} \sim 0$

$$\nabla \bullet \boldsymbol{j}^{MHD} \sim 0$$

$$\boldsymbol{j}^{MHD} \equiv ne\left(\boldsymbol{u}^p - \boldsymbol{u}^e\right)$$

Charge conservation equation in MHD

6) Equation of state

$P^p = n\, k_B\, T^p$ ... proton's equation of state

$P^e = n\, k_B\, T^e$ ... electron's equation of state

summation

$$P^{MHD} = 2\, n\, k_B\, T^{MHD} = 2\, \frac{\rho^{MHD}}{M+m} k_B\, T^{MHD} = \rho^{MHD} \frac{k_B}{\frac{M+m}{2}} T^{MHD} = \rho^{MHD} \frac{k_B}{\bar{m}} T^{MHD}$$

$T^p \sim T^e \sim T^{MHD}$

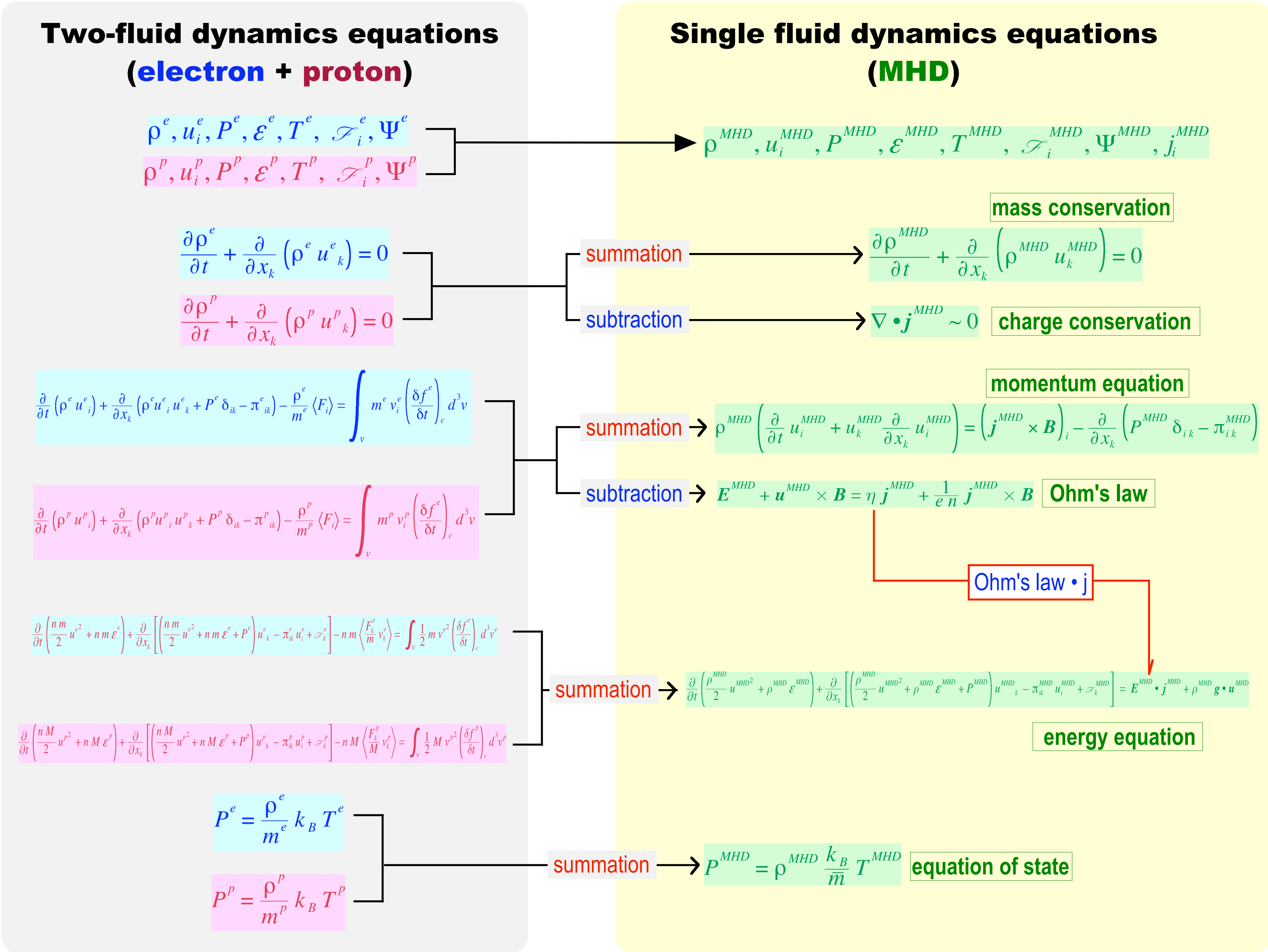
$n = \frac{\rho^{MHD}}{M+m}$

$\bar{m} \equiv \frac{M+m}{2}$

average mass

$$P^{MHD} = \rho^{MHD} \frac{k_B}{\bar{m}} T^{MHD}$$

Equation of state in MHD



Two-fluid dynamics equations
(electron + proton)
Single fluid dynamics equations
(MHD)
mass conservation
summation
subtraction
charge conservation
momentum equation
summation
subtraction
Ohm's law
Ohm's law • j
summation
energy equation
summation
equation of state

Electromagnetic part
(Maxwell's equations)

- Ampere's law
- magnetic flux conservation
- electromagnetic induction equation
- Coulmob's law
  - Ohm's law

(**Fluid** => **Field**)

Maxwell's equations (determine electromagnetic fields from the state of plasma)

MKS unit

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$ **... Generalized Ampere's law**

$$\nabla \bullet \boldsymbol{B} = 0$$ **... Conservation of magnetic flux**

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$ **... Electromagnetic induction equation (Faraday's law)**

$$\nabla \bullet \boldsymbol{E} = \frac{\rho_c}{\varepsilon_0}$$ **... Coulomb's law**

$$\varepsilon_0 = 8.854 \times 10^{-12}\ \mathrm{F\ m^{-1}}$$ **... permittivity of free space**

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\ m^{-1}}$$ **... magnetic permeability**

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.998 \times 10^{8}\ \mathrm{m\ s^{-1}}$$ **... speed of light**

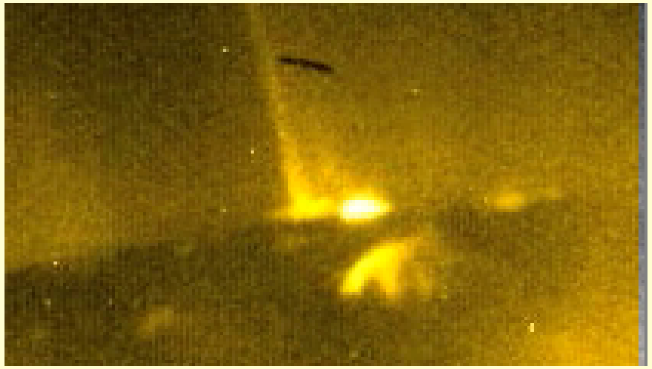
Units:

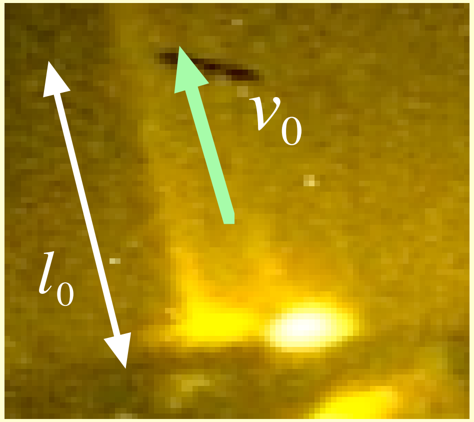
E... V/m, B... T (= 10^4 G), j... A/m^2

Typical scales in MHD

MHD phenomena… large-scale, slowly evolving phenomena

e.g. solar coronal jet





Typical scale:

length… $l_0 \sim$ 10^4 km

velocity… $v_0 \sim$ 100 km/s

time… $t_0 = \dfrac{l_0}{v_0} \sim$ 100 s

r_B (gyration radius for a proton) ~ 1 m
t_B (gyration period for a proton) ~ 10^{-4} s

Approximations used in MHD

• $v_0 << c$ **(non-relativistic approximation)**

Comparison between electromagnetic terms in the generalized Ampere's law

$$\boldsymbol{j} = \frac{1}{\mu_0}\left(\nabla\times\boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}\right)$$

$$\frac{B_0}{l_0} \qquad \frac{1}{c^2}\frac{E_0}{t_0}$$

Faraday's law

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \rightarrow \frac{E_0}{l_0} \sim \frac{B_0}{t_0}$$

$$\frac{1}{c^2}\frac{E_0}{t_0} \approx \frac{1}{c^2}\frac{1}{t_0}\frac{l_0}{t_0}B_0 = \frac{1}{c^2}\frac{l_0^2}{t_0^2}\frac{B_0}{l_0} = \frac{v_0^2}{c^2}\frac{B_0}{l_0} << \frac{B_0}{l_0}$$

$$\therefore \frac{B_0}{l_0} >> \frac{1}{c^2}\frac{E_0}{t_0} \Rightarrow we\ neglect\ \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

Electromagnetic wave is <u>NOT described</u> by MHD.

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \Rightarrow \nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j}$$

... Ampere's law in MHD

(determines magnetic field from current; consistent with **MHD charge conservation eq.**)