2) Momentum equation

mass conservation: $\frac{\partial \rho^\alpha}{\partial t} + \frac{\partial}{\partial x_k}\left(\rho^\alpha u^\alpha_k\right) = 0$

$$\frac{\partial}{\partial t}\left(\rho^\alpha u^\alpha_i\right) + \frac{\partial}{\partial x_k}\left(\rho^\alpha u^\alpha_i u^\alpha_k + P^\alpha \delta_{ik} - \pi^\alpha_{ik}\right) - \rho^\alpha \frac{\langle F^\alpha_i \rangle}{m^\alpha} = \int_v m^\alpha v^\alpha_i \left(\frac{\delta f^\alpha}{\delta t}\right)_c d^3 v^\alpha$$

$$\rho^\alpha \left(\frac{\partial u^\alpha_i}{\partial t} + \underline{u^\alpha_k \frac{\partial}{\partial x_k} u^\alpha_i}\right)$$

nonlinear term

$$\langle F^\alpha \rangle = \langle q^\alpha E + q^\alpha v \times B \rangle = q^\alpha E + q^\alpha u^\alpha \times B$$

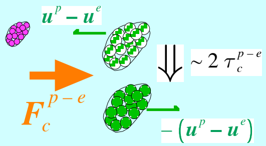
$\frac{\partial}{\partial t}, \frac{\partial}{\partial x_k}$ ... species-independent

$\frac{d^\alpha}{dt} \equiv \frac{\partial}{\partial t} + u^\alpha_k \frac{\partial}{\partial x_k}$ ... species-dependent

Firstly, we neglet nonlinear term & viscous stress for simplicity.

proton's momentum equation:

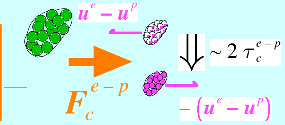
$$M\,n \frac{\partial \boldsymbol{u}^p}{\partial t} + \nabla P^p - en\left(\boldsymbol{E} + \boldsymbol{u}^p \times \boldsymbol{B}\right) = \int_v M\, v^p_i \left(\frac{\delta f^p}{\delta t}\right)_c d^3 v^p \Rightarrow \boldsymbol{F}^{p-e}_c = -M\,n\,\frac{\boldsymbol{u}^p - \boldsymbol{u}^e}{\tau^{p-e}_c}$$



momentum change per unit time = collisional force

electron's momentum equation:

$$m\,n \frac{\partial \boldsymbol{u}^e}{\partial t} + \nabla P^e + en\left(\boldsymbol{E} + \boldsymbol{u}^e \times \boldsymbol{B}\right) = \int_v m\, v^e_i \left(\frac{\delta f^e}{\delta t}\right)_c d^3 v^e \Rightarrow \boldsymbol{F}^{e-p}_c = -m\,n\,\frac{\boldsymbol{u}^e - \boldsymbol{u}^p}{\tau^{e-p}_c}$$



Add both equations...

$$n \frac{\partial}{\partial t}\left(M\,\boldsymbol{u}^p + m\,\boldsymbol{u}^e\right) = -\nabla\left(P^p + P^e\right) + e\,n\left(\boldsymbol{u}^p - \boldsymbol{u}^e\right) \times \boldsymbol{B}$$

$$\rho^{MHD} \frac{\partial}{\partial t} \boldsymbol{u}^{MHD} = -\nabla P^{MHD} + \boldsymbol{j}^{MHD} \times \boldsymbol{B}$$

$P^{MHD} \equiv P^p + P^e$: *total gas pressure*

$\boldsymbol{j}^{MHD} \equiv e\,n\left(\boldsymbol{u}^p - \boldsymbol{u}^e\right)$: *current density*

Simplified momentum eq. in MHD

$$\boldsymbol{F}^{p-e}_c + \boldsymbol{F}^{e-p}_c = \boldsymbol{0} \quad \left(\tau^{e-p}_c = \frac{m}{M}\tau^{p-e}_c\right)$$

$$\int_v M\, v^p_i \left(\frac{\delta f^p}{\delta t}\right)_c d^3 v^p + \int_v m\, v^e_i \left(\frac{\delta f^e}{\delta t}\right)_c d^3 v^e = 0$$

Conservation of total momentum via $^{p-e}_{e-p}$ collision

* Nonlinear term & viscous term

Nonlinear term:

$$\left(\rho^{p} u^{p}{}_{k}\right) \frac{\partial}{\partial x_k} u^{p}{}_{i} + \left(\rho^{e} u^{e}{}_{k}\right) \frac{\partial}{\partial x_k} u^{e}{}_{i} = n\, M \left(u^{p}{}_{k} \frac{\partial}{\partial x_k} u^{p}{}_{i} + \frac{m}{M} u^{e}{}_{k} \frac{\partial}{\partial x_k} u^{e}{}_{i} \right) = n\, M \left[u^{p}{}_{k} \frac{\partial}{\partial x_k} u^{p}{}_{i} \left(\frac{m}{M}\right)^{0} + O\left(\left(\frac{m}{M}\right)^{1}\right) \right]$$

1st-order term of $\frac{m}{M}$

$$\left(\rho^{MHD} u^{MHD}{}_{k}\right) \frac{\partial}{\partial x_k} u^{MHD}{}_{i} = n\,(M+m) \frac{M\, u^{p}{}_{k} + m\, u^{e}{}_{k}}{M+m} \frac{\partial}{\partial x_k} \left(\frac{M\, u^{p}{}_{i} + m\, u^{e}{}_{i}}{M+m} \right)$$

$$= n\, M \left(u^{p}{}_{k} + \frac{m}{M} u^{e}{}_{k} \right) \frac{\partial}{\partial x_k} \left(\frac{u^{p}{}_{i} + \frac{m}{M} u^{e}{}_{i}}{1 + \frac{m}{M}} \right) = n\, M\, u^{p}{}_{k} \frac{\partial}{\partial x_k} u^{p}{}_{i} \left(\frac{m}{M}\right)^{0} + O\left(\left(\frac{m}{M}\right)^{1}\right)$$

1st- and higher-order terms of $\frac{m}{M}$

$$\left(\rho^{p} u^{p}{}_{k}\right) \frac{\partial}{\partial x_k} u^{p}{}_{i} + \left(\rho^{e} u^{e}{}_{k}\right) \frac{\partial}{\partial x_k} u^{e}{}_{i} \approx \left(\rho^{MHD} u^{MHD}{}_{k}\right) \frac{\partial}{\partial x_k} u^{MHD}{}_{i}$$

based on the 0th-order approximation of $\frac{m}{M}$

Viscous term:

$$\frac{\partial \pi^{p}{}_{ik}}{\partial x_k} + \frac{\partial \pi^{e}{}_{ik}}{\partial x_k} = \frac{\partial}{\partial x_k}\left(\mu^{p} D^{p}_{ik} + \mu^{e} D^{e}_{ik}\right) = \frac{\partial}{\partial x_k}\left[\mu^{p}\left(D^{p}_{ik} + \frac{\mu^{e}}{\mu^{p}} D^{e}_{ik}\right)\right]$$
$$= \frac{\partial}{\partial x_k}\left[\mu^{p}\left(D^{p}_{ik} + \sqrt{\frac{m}{M}}\, D^{e}_{ik}\right)\right]$$

$$\frac{\partial \pi^{MHD}{}_{ik}}{\partial x_k} = \frac{\partial}{\partial x_k}\left(\mu^{MHD} D^{MHD}_{ik}\right) = \frac{\partial}{\partial x_k}\left[\mu^{MHD}\left\{\frac{\partial\left(\dfrac{u^{p}{}_{i} + \frac{m}{M} u^{e}{}_{i}}{1+\frac{m}{M}}\right)}{\partial x_k} + \frac{\partial\left(\dfrac{u^{p}{}_{k} + \frac{m}{M} u^{e}{}_{k}}{1+\frac{m}{M}}\right)}{\partial x_i} - \frac{2}{3}\nabla\bullet\left(\frac{\boldsymbol{u}^{p} + \frac{m}{M}\boldsymbol{u}^{e}}{1+\frac{m}{M}}\right)\delta_{ik}\right\}\right]$$

Here we use $\mu^{MHD} \sim \mu^{p} \gg \mu^{e}$ because viscosity is proportional to $\sqrt{m^{\alpha}}$.

$\alpha = proton, electron$

$$\frac{\partial \pi^{p}{}_{ik}}{\partial x_k} + \frac{\partial \pi^{e}{}_{ik}}{\partial x_k} \approx \frac{\partial \pi^{MHD}{}_{ik}}{\partial x_k}$$

based on the 0th-order approximation of $\frac{m}{M}$

$$\rho^{MHD}\left(\frac{\partial}{\partial t} u_i^{MHD} + u_k^{MHD}\frac{\partial}{\partial x_k} u_i^{MHD}\right) = -\frac{\partial}{\partial x_k}\left(P^{MHD}\,\delta_{ik} - \pi^{MHD}_{ik}\right) + \left(\boldsymbol{j}^{MHD} \times \boldsymbol{B}\right)_i$$

Momentum eq. in MHD