Chapman-Enskog procedure

In general, π_{ik} and $\mathscr{F}_i$ are expressed using spatial derivatives of **u** & T.

$$\pi_{ik} = \mu D_{ik} \qquad D_{ik} \equiv \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}(\nabla \bullet \boldsymbol{u})\,\delta_{ik} \ \text{(deformation rate tensor)}$$

... expressed using spatial derivative of **u**

$$\Psi \equiv \pi_{ik}\frac{\partial u_i}{\partial x_k} = \frac{1}{2}\mu \| D \|^2 \geq 0 \ \text{(always)}$$

... expressed using spatial derivative of **u**

$$\| D \|^2 \equiv D_{ik} D_{ik}$$

$$\mathscr{F}_i \equiv \rho \left\langle \frac{1}{2} |\boldsymbol{w}|^2 w_i \right\rangle = -\kappa \frac{\partial T}{\partial x_i}$$

... expressed using spatial derivative of T

This determines **diffusive fluid system** governed by **Navier-Stokes equations**:

Navier-Stokes equations ... 5 equations with 14 quantities $\rho, u_i, P, T, \pi_{ik}, \mathscr{F}_k$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k + P\,\delta_{ik} - \pi_{ik}) = \rho \frac{\langle F_i \rangle}{m}$$

$$\frac{\partial}{\partial t}(\rho \mathcal{E}) + \frac{\partial}{\partial x_k}(\rho \mathcal{E} u_k) = -P\frac{\partial u_k}{\partial x_k} - \frac{\partial \mathscr{F}_k}{\partial x_k} + \Psi$$

$$\rho \mathcal{E} \equiv \frac{1}{2}\rho \langle w_i w_i \rangle = \frac{1}{2}\rho \left\langle |\boldsymbol{w}|^2 \right\rangle = \frac{3}{2}P \qquad \Psi \equiv \pi_{ik}\frac{\partial u_i}{\partial x_k}$$

Closure relation $\langle w_i^2 \rangle_f \sim \langle w_i^2 \rangle_{f_{MW}} = \frac{k_B}{m}T$

$$P = \frac{\rho}{m} k_B T \qquad \pi_{ik} = \mu D_{ik} \qquad \mathscr{F}_i = -\kappa \frac{\partial T}{\partial x_i}$$

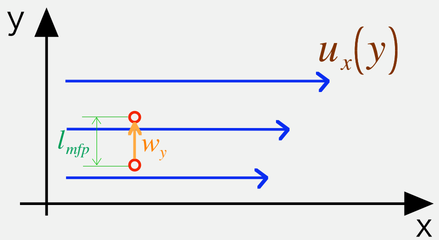
9 equations

Order estimate of diffusion coefficients μ & κ (based on physical interpretation)

• **Viscosity** μ $(\equiv \rho\nu\ ;\ \nu$: kinematic viscosity$)$

$$\pi_{xy} \equiv -\rho\left\langle w_x w_y \right\rangle \sim -\rho\,(\Delta u_x)\, w_y = -n\, m\left(-\frac{\partial u_x}{\partial y}\, l_{mfp}\right) v_T = \mu\,\frac{\partial u_x}{\partial y}$$

transport of x-comp. of momentum in y-direction via collision



l_{mfp}: *mean free path*

$$v_T \sim \sqrt{\frac{k_B\,T}{m}}$$

$$\mu \sim n\, l_{mfp}\sqrt{m\, k_B\, T} \quad \longrightarrow \quad \propto \sqrt{m}$$

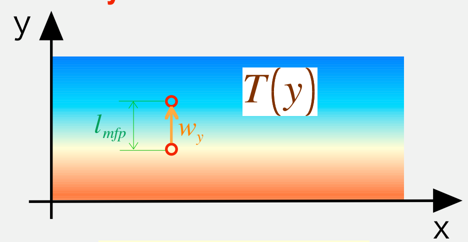
$$\nu \sim l_{mfp}\, v_T$$

• **Thermal conductivity** κ $(\equiv \rho C_p \alpha\ ;\ \alpha$: thermal diffusivity$)$

$$\mathscr{T}_y \equiv \rho\left\langle \frac{1}{2}\,|\mathbf{w}|^2 w_y \right\rangle \sim \rho\left(\frac{3}{2}\,\frac{k_B}{m}\Delta T\right) w_y = n\, m\,\frac{3}{2}\,\frac{k_B}{m}\left(-\frac{\partial T}{\partial y}\, l_{mfp}\right) v_T$$

$$= -\kappa\,\frac{\partial T}{\partial y}$$

transport of internal energy in y-direction via collision



l_{mfp}: *mean free path*

$$v_T \sim \sqrt{\frac{k_B\,T}{m}}$$

$$\alpha \sim l_{mfp}\, v_T$$

$$\kappa \sim \frac{3}{2}\, n\, l_{mfp}\sqrt{\frac{k_B^3\,T}{m}} \quad \longrightarrow \quad \propto 1/\sqrt{m}$$

$$\frac{\kappa}{\mu} \sim \frac{3}{2}\,\frac{k_B}{m} = C_V$$

specific heat at constant volume

Classification of dynamic states

Classification of dynamic states based on collision time scale t_c

$t_{phenomenon} \ll t_c$... f **is far from** f_{MW} $L \ll l_{mfp}$

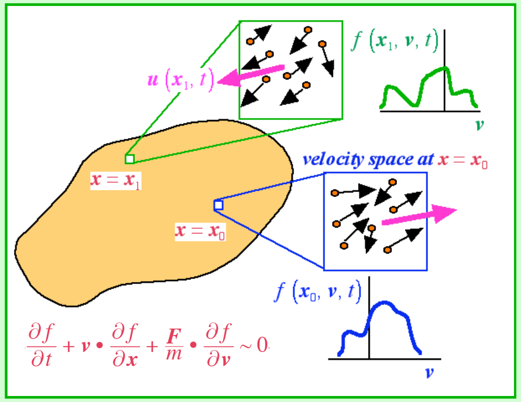
$t_{phenomenon}$: typical time scale of a phenomenon
L: typical length scale of a phenomenon

- **Collision** is **less frequent**
- **Nonthermal state**... temperature cannot be determined

  f is determined by **collisionless Boltzmann's equation**:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \bullet \frac{\partial f}{\partial \boldsymbol{x}} + \frac{\boldsymbol{F}}{m} \bullet \frac{\partial f}{\partial \boldsymbol{v}} \sim 0$$

collisionless state



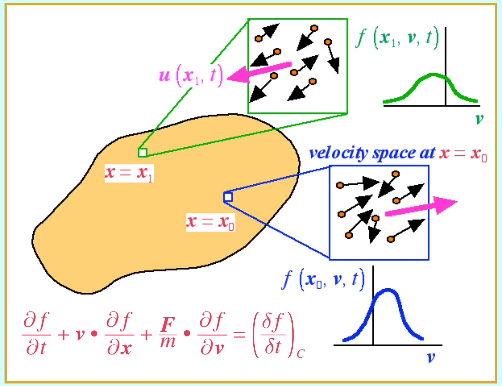
$t_{phenomenon} \sim t_c$... f **is relaxing to** f_{MW} $L \sim l_{mfp}$

- **Collision** is **frequent**
- **Nonthermal** or **thermal state**

  f is determined by **collisional Boltzmann's equation**:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \bullet \frac{\partial f}{\partial \boldsymbol{x}} + \frac{\boldsymbol{F}}{m} \bullet \frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\delta f}{\delta t}\right)_c$$

collisional state



$t_{phenomenon} \gg t_c$... f **is close to** f_{MW} $L \gg l_{mfp}$

- **Collision** is **very frequent**
- **Thermal state**... temperature can be determined

thermal state

Fluid dynamics equations: