1) Collisional equilibrium case

In this case collision does NOT change the distribution function (**collisional equilibrium**). In a local frame fixed to ΔV at $\boldsymbol{x} = \boldsymbol{x}_0$, it is given by

$$\left(\frac{\delta f\left(\boldsymbol{x}_0, \boldsymbol{v}, t\right)}{\delta t}\right)_C = 0 \quad \text{with } \boldsymbol{v} = \boldsymbol{w}$$

$$\left(\frac{\delta f\left(\boldsymbol{x}_0, \boldsymbol{v}, t\right)}{\delta t}\right)_C = \int_{V_v, \Omega} f\left(\boldsymbol{x}_0, \boldsymbol{v}'\right) f\left(\boldsymbol{x}_0, \boldsymbol{v}_1'\right) \sigma_{\boldsymbol{v}', \boldsymbol{v}_1' \to \boldsymbol{v}, \boldsymbol{v}_1}\left(\Omega'\right) \left|\boldsymbol{v}' - \boldsymbol{v}_1'\right| d\Omega' \, d^3 v_1' - f\left(\boldsymbol{x}_0, \boldsymbol{v}\right) f\left(\boldsymbol{x}_0, \boldsymbol{v}_1\right) \sigma_{\boldsymbol{v}, \boldsymbol{v}_1 \to \boldsymbol{v}', \boldsymbol{v}_1'}(\Omega) \left|\boldsymbol{v} - \boldsymbol{v}_1\right| d\Omega \, d^3 v_1$$

existence probability of a particle with $\boldsymbol{v}_1'$

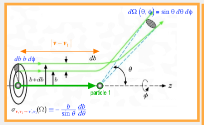
existence probability of a particle with $\boldsymbol{v}'$

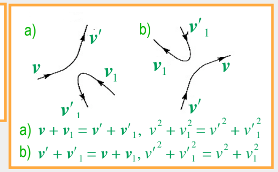
existence probability of a particle with $\boldsymbol{v}_1$

existence probability of a particle with $\boldsymbol{v}$

occurrence probability of a collisional process $\boldsymbol{v}', \boldsymbol{v}_1' \to \boldsymbol{v}, \boldsymbol{v}_1$
=
occurrence probability of a collisonal process $\boldsymbol{v}, \boldsymbol{v}_1 \to \boldsymbol{v}', \boldsymbol{v}_1'$

$$\sigma_{\boldsymbol{v}', \boldsymbol{v}_1' \to \boldsymbol{v}, \boldsymbol{v}_1}\left(\Omega'\right) \left|\boldsymbol{v}' - \boldsymbol{v}_1'\right| d\Omega' \, d^3 v_1'$$
$$= \sigma_{\boldsymbol{v}, \boldsymbol{v}_1 \to \boldsymbol{v}', \boldsymbol{v}_1'}(\Omega) \left|\boldsymbol{v} - \boldsymbol{v}_1\right| d\Omega \, d^3 v_1$$





occurrence rate of $\boldsymbol{v}'$ => $\boldsymbol{v}$ — *occurrence rate of $\boldsymbol{v}$ => $\boldsymbol{v}'$*

$$\left[\frac{\delta f\left(\boldsymbol{x}_0, \boldsymbol{v}, t\right)}{\delta t}\right]_C = 0 \quad \Leftrightarrow \quad f\left(\boldsymbol{x}_0, \boldsymbol{v}'\right) f\left(\boldsymbol{x}_0, \boldsymbol{v}_1'\right) - f\left(\boldsymbol{x}_0, \boldsymbol{v}\right) f\left(\boldsymbol{x}_0, \boldsymbol{v}_1\right) = 0$$

When the distribution function depends on the magnitude of velocity, but does not depend on its direction (isotropic random motion),

$$f(\boldsymbol{v}) \Rightarrow f\left(v^2\right), \; f\left(\boldsymbol{v}'\right) f\left(\boldsymbol{v}_1'\right) - f(\boldsymbol{v}) f(\boldsymbol{v}_1) = 0 \Rightarrow f\left(v'^2\right) f\left(v_1'^2\right) - f\left(v^2\right) f\left(v_1^2\right) = 0$$
$$f\left(v'^2\right) f\left(v_1'^2\right) - f\left(v^2\right) f\left(v_1^2\right) = 0, \; v^2 + v_1^2 = v'^2 + v_1'^2 \Rightarrow f\left(v^2\right) = C_0 \, e^{-\alpha v^2}$$

This gives **Maxwellian distribution function** $f_{MW}(v)$ with $\int_v f_{MW} \, d^3 v = n$ and $\frac{1}{2} m \left\langle |\boldsymbol{w}|^2 \right\rangle_{MW} = \frac{3}{2} k_B T$:

$$f_{MW}(v) \equiv n \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} e^{-\frac{m|\boldsymbol{v}|^2}{2 k_B T}}$$

Relation between **temperature** and **random motion**
⇓
Definition of **temperature**

Gas pressure and internal energy density are then expressued using temperature (here we use $\boldsymbol{v} = \boldsymbol{w}$):

$$P \equiv \frac{1}{3} n m \left\langle |\boldsymbol{w}|^2 \right\rangle_{MW} = n k_B T$$

... **equation of state** is derived from **Maxwellian distribution**

$$\rho \, \mathcal{E} \equiv \frac{1}{2} n m \left\langle |\boldsymbol{w}|^2 \right\rangle_{MW} = \frac{3}{2} n k_B T$$

When $f = f_{MW}$, diffusion terms disappear... **no diffusion*** in the **collisional equilibrium case**

$$f = f_{MW} \longrightarrow \begin{cases} \pi_{ik} = 0 \;\longrightarrow\; \Psi = 0 \\ \mathscr{F}_k \equiv \rho \left\langle \frac{1}{2} |\boldsymbol{w}|^2 w_k \right\rangle = 0 \end{cases}$$

*Isotropic random motion ($\langle w_1^2 \rangle = \langle w_2^2 \rangle = \langle w_3^2 \rangle$) is assumed.

When a locally uniform average flow exists, the Maxwellian distribution is given by (via **Galilei transformation** ($\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}, d^3v = d^3w$))

$$f_{MW}(v) = n \left(\frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} e^{-\frac{m\left[(v_x - u_x)^2 + (v_y - u_y)^2 + (v_z - u_z)^2\right]}{2 k_B T}} \longrightarrow \langle v \rangle = \frac{\int (\boldsymbol{u} + \boldsymbol{w})\, f_{MW}(w)\, d^3w}{\int f_{MW}(w)\, d^3w} = \boldsymbol{u}$$

When $f = f_{MW}$ at every ΔV and each time, it gives **ideal fluid system** governed by **Euler equations**:

Euler equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_k}(\rho u_i u_k + P \delta_{ik}) = \rho \frac{\langle F_i \rangle}{m}$$

$$\frac{\partial}{\partial t}(\rho \varepsilon) + \frac{\partial}{\partial x_k}(\rho \varepsilon u_k) = -P \frac{\partial u_k}{\partial x_k}$$

$$\rho \varepsilon \equiv \frac{1}{2} \rho \langle w_i w_i \rangle = \frac{1}{2} \rho \left\langle |\boldsymbol{w}|^2 \right\rangle = \frac{3}{2} P$$

... 5 equations for 6 quantities ρ, u_i, P, T

closure relation

$$P = \frac{\rho}{m} k_B T$$

... 1 equation