$F_k\, v_k = 0$ for *Lorentz force*

$\left\langle \frac{F_k}{m} v_k \right\rangle = \frac{F_k}{m}\langle v_k\rangle = \frac{F_k}{m} u_k$ for *gravitational force, Coulomb force*

total energy equation

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}|\boldsymbol{u}|^2 + \rho\,\mathcal{E}\right) + \frac{\partial}{\partial x_k}\left(\left[\frac{\rho}{2}|\boldsymbol{u}|^2 + \rho\,\mathcal{E} + P\right]u_k - \pi_{ik}\,u_i + \mathscr{F}_k\right) = \rho\left\langle \frac{F_k}{m} v_k \right\rangle$$

rate of work done by external force

flow energy density

internal energy density

total energy flux (energy transported by average flow)

enthalpy

viscous flux

conduction flux (internal energy transported by random motion)

Internal energy equation is derived by subtracting **flow energy equation** from the **total energy equation**:

momentum equation • $\boldsymbol{u}$: (inner product)

$$\frac{\partial}{\partial t}(\rho\, u_i) + \frac{\partial}{\partial x_k}(\rho\, u_i\, u_k + P\,\delta_{ik} - \pi_{ik}) = \rho\frac{\langle F_i\rangle}{m} \quad \times u_i$$

continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho\, u_k) = 0$$

$\langle \boldsymbol{F}\rangle = q\,\langle \boldsymbol{v}\rangle \times \boldsymbol{B} = q\,\boldsymbol{u}\times\boldsymbol{B} \;\Rightarrow\; \langle F_i\rangle\, u_i = 0$ for *Lorentz force*

$\langle F_i\rangle\, u_i = F_i\, u_i$ for *gravitational force, Coulomb force*

flow energy equation

subtract

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}|\boldsymbol{u}|^2\right) + \frac{\partial}{\partial x_k}\left(\frac{\rho}{2}|\boldsymbol{u}|^2 u_k\right) = \rho\frac{\langle F_i\rangle}{m}u_i - u_i\frac{\partial P}{\partial x_i} + u_i\frac{\partial \pi_{ik}}{\partial x_k}$$

rate of work done by external force

rate of work done by gas pressure gradient force

rate of work done by viscous force

internal energy equation

$$\frac{\partial}{\partial t}\rho\,\mathcal{E} + \frac{\partial}{\partial x_k}\rho\,\mathcal{E}\,u_k = -P\frac{\partial u_k}{\partial x_k} - \frac{\partial \mathscr{F}_k}{\partial x_k} + \Psi$$

$$\Psi \equiv \pi_{ik}\frac{\partial u_i}{\partial x_k} \quad \text{(viscous dissipation rate)}$$

$dU \qquad -p\,dV \qquad \delta Q$ ... 1st law of thermodynamics

Fluid dynamics equations for single species system:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho\, u_k) = 0$$

$$\frac{\partial}{\partial t}(\rho\, u_i) + \frac{\partial}{\partial x_k}(\rho\, u_i\, u_k + P\,\delta_{ik} - \pi_{ik}) = \rho\,\frac{\langle F_i\rangle}{m}$$

$$\frac{\partial}{\partial t}\left(\rho\,\mathcal{E}\right) + \frac{\partial}{\partial x_k}\left(\rho\,\mathcal{E}\,u_k\right) = -P\,\frac{\partial u_k}{\partial x_k} - \frac{\partial \mathscr{F}_k}{\partial x_k} + \Psi$$

... 5 equations

$$\rho\,\mathcal{E} \equiv \frac{1}{2}\,\rho\,\langle w_i\, w_i\rangle = \frac{1}{2}\,\rho\,\langle |\mathbf{w}|^2\rangle = \frac{3}{2}\,P \qquad \Psi \equiv \pi_{ik}\,\frac{\partial u_i}{\partial x_k}$$

Number of quantities: ρ, u_i, P, π_{ik} (symmetric tensor, 5 unknowns), $\mathscr{F}_k$

$$\pi_{ik} = \begin{pmatrix} \frac{1}{3}\,\rho\left(-2\,\langle w_1^2\rangle + \langle w_2^2\rangle + \langle w_3^2\rangle\right) & -\rho\,\langle w_1 w_2\rangle & -\rho\,\langle w_1 w_3\rangle \\ -\rho\,\langle w_2 w_1\rangle & \frac{1}{3}\,\rho\left(\langle w_1^2\rangle - 2\,\langle w_2^2\rangle + \langle w_3^2\rangle\right) & -\rho\,\langle w_2 w_3\rangle \\ -\rho\,\langle w_3 w_1\rangle & -\rho\,\langle w_3 w_2\rangle & \boxed{\frac{1}{3}\,\rho\left(\langle w_1^2\rangle + \langle w_2^2\rangle - 2\,\langle w_3^2\rangle\right)} \end{pmatrix}$$

... 13 quantities

$$\pi_{33} = -\left(\pi_{11} + \pi_{22}\right)$$

Additional constraints are needed => **closure relation**

Closure relation

$$f \longrightarrow f_{MW}$$

So far we have not yet assumed any specific distribution function.

$$P \equiv \frac{1}{3}\rho\left(\langle w_1^2\rangle + \langle w_2^2\rangle + \langle w_3^2\rangle\right) = \frac{1}{3}\rho\left\langle |\boldsymbol{w}|^2\right\rangle$$

$$\pi_{ik} \equiv \begin{pmatrix} \frac{1}{3}\rho\left(-2\langle w_1^2\rangle + \langle w_2^2\rangle + \langle w_3^2\rangle\right) & -\rho\langle w_1 w_2\rangle & -\rho\langle w_1 w_3\rangle \\ -\rho\langle w_2 w_1\rangle & \frac{1}{3}\rho\left(\langle w_1^2\rangle - 2\langle w_2^2\rangle + \langle w_3^2\rangle\right) & -\rho\langle w_2 w_3\rangle \\ -\rho\langle w_3 w_1\rangle & -\rho\langle w_3 w_2\rangle & \frac{1}{3}\rho\left(\langle w_1^2\rangle + \langle w_2^2\rangle - 2\langle w_3^2\rangle\right) \end{pmatrix}$$

$$\langle \boxed{\diamond} \rangle = \frac{\int_v \boxed{\diamond}\, f\, d^3v}{\int_v f\, d^3v}$$

$$\rho\,\mathcal{E} \equiv \frac{1}{2}\rho\,\langle w_i w_i\rangle = \frac{1}{2}\rho\left\langle |\boldsymbol{w}|^2\right\rangle = \frac{3}{2}P$$

$$\mathscr{F}_k \equiv \frac{1}{2}\rho\,\langle w_i w_i w_k\rangle = \frac{1}{2}\rho\left\langle |\boldsymbol{w}|^2 w_k\right\rangle$$

To derive the closure relation, we assume a specific distribution function, which is Maxwellian distribution function $f_{MW}\left(\boldsymbol{x}, \boldsymbol{w}(T(\boldsymbol{x}, t)), t\right)$.

$$f(\boldsymbol{x}, \boldsymbol{w}, t) \longrightarrow f_{MW}\left(\boldsymbol{x}, \boldsymbol{w}(T(\boldsymbol{x}, t)), t\right)$$