For single species system, $\int_v \chi(\mathbf{v}) \left(\frac{\delta f}{\delta t}\right)_c d^3 v = 0 \quad when\ \chi(\mathbf{v}) = m, m\mathbf{v}, \frac{1}{2}mv^2$ **(all are conserved)**

$$\frac{\partial}{\partial t}\left(n \langle \chi \rangle\right) + \frac{\partial}{\partial x_k}\left(n \langle \chi\, v_k \rangle\right) - \left(n \left\langle \frac{F_k}{m} \frac{\partial \chi}{\partial v_k} \right\rangle\right) = 0 \quad \ldots ※$$

1. Mass conservation

When $\chi = m$, then $n\langle m \rangle = \rho,\ n\langle m\, v_k \rangle = \rho\, u_k,\ n\left\langle \frac{F_k}{m} \frac{\partial m}{\partial v_k} \right\rangle = 0$

※ => $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}\left(\rho\, u_k\right) = 0$$

"continuity equation"

2. Momentum conservation

When $\chi = m v_i$, then $n\langle m\, v_i \rangle = \rho\, u_i,\ n\langle m\, v_i\, v_k \rangle = \rho \langle v_i\, v_k \rangle, n\left\langle \frac{F_k}{m} \frac{\partial (m v_i)}{\partial v_k} \right\rangle = n\left\langle \frac{F_k}{m} m\, \delta_{ik} \right\rangle$

※ => $$\frac{\partial}{\partial t}\left(\rho\, u_i\right) + \frac{\partial}{\partial x_k}\left(\rho \langle v_i v_k \rangle\right) - \rho \frac{\langle F_i \rangle}{m} = 0$$

"momentum equation"

About $\langle v_i v_k \rangle$...

$$\langle v_i v_k \rangle = \langle (u_i + w_i)(u_k + w_k) \rangle$$
$$= \langle u_i u_k \rangle + u_i \langle w_k \rangle + u_k \langle w_i \rangle + \langle w_i w_k \rangle$$
$$= u_i u_k + \underbrace{\langle w_i w_k \rangle}_{\text{random motion contribution}}$$

(because $\langle w_k \rangle = \langle w_i \rangle = 0$)

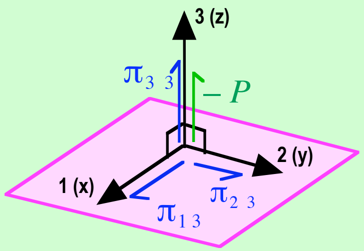
k, h... *function of* $(\boldsymbol{x}, t)$

$$\langle k\chi + h\psi \rangle = \frac{\int (k\chi + h\psi) f \, d^3v}{\int f \, d^3v} = \frac{k \int \chi f \, d^3v}{\int f \, d^3v} + \frac{h \int \psi f \, d^3v}{\int f \, d^3v} = k\langle \chi \rangle + h \langle \psi \rangle$$

$\rho \langle w_i w_k \rangle$ is decomposed into **isotropic part** and **non-isotropic part**:

$$\sigma_{ik} \equiv -\rho \langle w_i w_k \rangle = -P\,\delta_{ik} + \pi_{ik}$$

stress tensor   gas pressure   viscous stress



P, π_{ik}... *surface force per unit area*

$$P\,\delta_{ik} \equiv \begin{pmatrix} \frac{1}{3}\rho\left(\langle w_1^2\rangle + \langle w_2^2\rangle + \langle w_3^2\rangle\right) & 0 & 0 \\ 0 & \frac{1}{3}\rho\left(\langle w_1^2\rangle + \langle w_2^2\rangle + \langle w_3^2\rangle\right) & 0 \\ 0 & 0 & \frac{1}{3}\rho\left(\langle w_1^2\rangle + \langle w_2^2\rangle + \langle w_3^2\rangle\right) \end{pmatrix}$$

$$\pi_{ik} \equiv \begin{pmatrix} \frac{1}{3}\rho\left(-2\langle w_1^2\rangle + \langle w_2^2\rangle + \langle w_3^2\rangle\right) & -\rho\langle w_1 w_2\rangle & -\rho\langle w_1 w_3\rangle \\ -\rho\langle w_2 w_1\rangle & \frac{1}{3}\rho\left(\langle w_1^2\rangle - 2\langle w_2^2\rangle + \langle w_3^2\rangle\right) & -\rho\langle w_2 w_3\rangle \\ -\rho\langle w_3 w_1\rangle & -\rho\langle w_3 w_2\rangle & \frac{1}{3}\rho\left(\langle w_1^2\rangle + \langle w_2^2\rangle - 2\langle w_3^2\rangle\right) \end{pmatrix}$$

$P \equiv \frac{1}{3}\rho\left(\langle w_1^2\rangle + \langle w_2^2\rangle + \langle w_3^2\rangle\right) = \frac{1}{3}\rho\langle |\boldsymbol{w}|^2\rangle$ ... definition of **gas pressure**

when random motion is isotropic ($\langle w_1^2\rangle = \langle w_2^2\rangle = \langle w_3^2\rangle$),
$\pi_{11} = \pi_{22} = \pi_{33} = 0$

$$\frac{\partial}{\partial t}(\rho\, u_i) + \frac{\partial}{\partial x_k}\left(\rho\,\langle v_i v_k\rangle\right) - \rho\,\frac{\langle F_i\rangle}{m} = 0$$

$$\frac{\partial}{\partial t}(\rho\, u_i) + \frac{\partial}{\partial x_k}\left(\underbrace{\rho\, u_i\, u_k}_{\text{flow momentum transported by average flow (macroscale effect)}} + \underbrace{P\,\delta_{ik} - \pi_{ik}}_{\text{random motion momentum transported by random motion (microscale effect)}}\right) = \rho\,\frac{\langle F_i\rangle}{m}$$

$P\,\delta_{ik}$...transports normal component of random motion momentum

$-\pi_{ik}$... transports tangential component of random motion momentum

3. Energy conservation

When $\chi = m\,\frac{v^2}{2} = \frac{1}{2}m\,v_i\,v_i = \frac{1}{2}m\,(u_i + w_i)\,(u_i + w_i)$

then

$$n\,\langle\chi\rangle = \rho\left(\frac{u_i\,u_i}{2} + \frac{\langle w_i\,w_i\rangle}{2}\right)$$

$$\rho\,\langle w_i w_k\rangle = P\,\delta_{ik} - \pi_{ik}$$

$$n\,\langle\chi v_k\rangle = \frac{\rho}{2}\left\langle (u_i + w_i)^2 (u_k + w_k)\right\rangle = \frac{\rho}{2}\left[u_i\,u_i\,u_k + 2\,u_i\,\langle w_i w_k\rangle + u_k\,\langle w_i\,w_i\rangle + \langle w_i\,w_i\,w_k\rangle\right]$$

$$n\left\langle\frac{F_k}{m}\,\frac{\partial\chi}{\partial v_k}\right\rangle = n\left\langle\frac{F_k}{m}\,\frac{\partial}{\partial v_k}\left(\frac{m}{2}\,v_i\,v_i\right)\right\rangle$$

$$= \rho\left\langle\frac{F_k}{m}\,v_i\,\delta_{ik}\right\rangle = \rho\left\langle\frac{F_k}{m}\,v_k\right\rangle$$

Definition of
conduction flux

$$\mathscr{F}_k \equiv \frac{1}{2}\,\rho\,\langle w_i\,w_i\,w_k\rangle = \frac{1}{2}\,\rho\,\left\langle |\boldsymbol{w}|^2\,w_k\right\rangle$$

transports internal energy via random motion

Definition of **internal energy density**

$$\rho\,\mathcal{E} \equiv \frac{1}{2}\,\rho\,\langle w_i\,w_i\rangle = \frac{1}{2}\,\rho\,\left\langle |\boldsymbol{w}|^2\right\rangle = \frac{3}{2}\,P$$