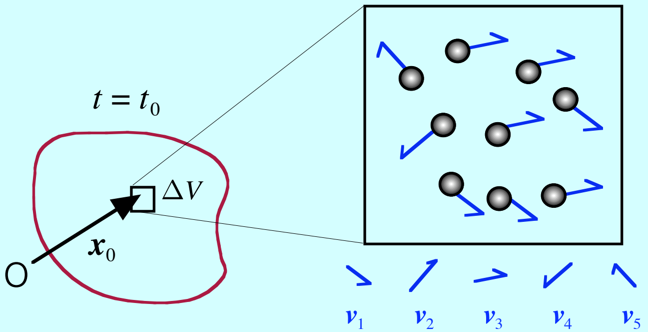
Derivation of fluid dynamics equations from Boltzmann's equation

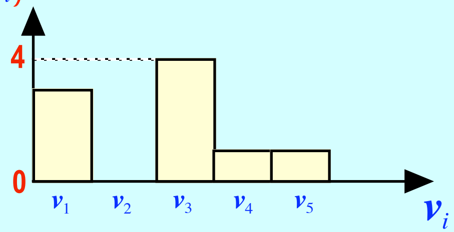
Distribution function

What is distribution function and how to use it to derive an average

When the velocity of a particle only takes specific values => **discrete distribution function**



$f(\boldsymbol{v}_i)$ (= $f(\boldsymbol{x}_0, \boldsymbol{v}_i, t_0)$) : distribution function (=> probability)



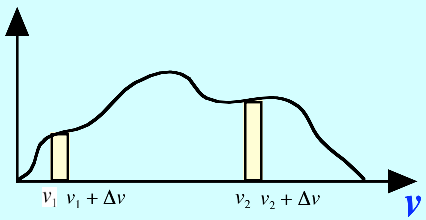
$$\langle \chi \rangle = \frac{\chi(\boldsymbol{v}_1)\, f(\boldsymbol{v}_1) + \chi(\boldsymbol{v}_2)\, f(\boldsymbol{v}_2) + \chi(\boldsymbol{v}_3)\, f(\boldsymbol{v}_3) + \chi(\boldsymbol{v}_4)\, f(\boldsymbol{v}_4) + \chi(\boldsymbol{v}_5)\, f(\boldsymbol{v}_5)}{f(\boldsymbol{v}_1) + f(\boldsymbol{v}_2) + f(\boldsymbol{v}_3) + f(\boldsymbol{v}_4) + f(\boldsymbol{v}_5)} = \frac{\sum_{i=1}^{5} \chi(\boldsymbol{v}_i)\, f(\boldsymbol{v}_i)}{\boxed{\sum_{i=1}^{5} f(\boldsymbol{v}_i)}}$$

= 3+0+4+1+1 = 9

... number of particles in ΔV

When the velocity of a particle takes any values => **continuous distribution function**

$f(\boldsymbol{v})$ (= $f(\boldsymbol{x}_0, \boldsymbol{v}, t_0)$)



$$\langle \chi \rangle = \frac{\int \chi(\boldsymbol{v})\, f(\boldsymbol{v})\, d\boldsymbol{v}}{\boxed{\int f(\boldsymbol{v})\, d\boldsymbol{v}}}$$

number density of particles at $\boldsymbol{x} = \boldsymbol{x}_0$

(= number of particles per unit volume at $\boldsymbol{x} = \boldsymbol{x}_0$)

Boltzmann's equation

Boltzmann's equation... describes the evolution of distribution function

Distribution function and number density:

$f\,(\boldsymbol{x}, \boldsymbol{v}, t)$... number density in position & velocity space (configuration space) at time t

$n(\boldsymbol{x}, t) = \int_v f(\boldsymbol{x}, \boldsymbol{v}, t)\, d^3 v$... number density in position space at time t

$N(t) = \int_x n(\boldsymbol{x}, t)\, d^3 x$... total number of particles at time t

Boltzmann's equation: $\dfrac{df}{dt} = \mathcal{F}_{collision} = \left(\dfrac{\delta f}{\delta t}\right)_C$

$$\frac{d\boldsymbol{v}}{dt} = \frac{\boldsymbol{F}}{m} \Leftrightarrow \frac{df}{dt} = \mathcal{F}_{collision}$$

Total differential of $f\,(\boldsymbol{x}, \boldsymbol{v}, t)$... $df = \dfrac{\partial f}{\partial \boldsymbol{x}} \bullet d\boldsymbol{x} + \dfrac{\partial f}{\partial \boldsymbol{v}} \bullet d\boldsymbol{v} + \dfrac{\partial f}{\partial t}\, dt$

time derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \boldsymbol{x}} \bullet \frac{d\boldsymbol{x}}{dt} + \frac{\partial f}{\partial \boldsymbol{v}} \bullet \frac{d\boldsymbol{v}}{dt} = \left(\frac{\delta f}{\delta t}\right)_C$$

Mechanical equation $\dfrac{d\boldsymbol{x}}{dt} = \boldsymbol{v}\,,\ \dfrac{d\boldsymbol{v}}{dt} = \dfrac{\boldsymbol{F}}{m}$

non-collisional force

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \bullet \frac{\partial f}{\partial \boldsymbol{x}} + \frac{\boldsymbol{F}}{m} \bullet \frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\delta f}{\delta t}\right)_C$$

$\left(\dfrac{\delta f}{\delta t}\right)_C$... Source term representing the net increase of $f\,(\boldsymbol{x}, \boldsymbol{v}, t)$ during δt via collision (collision term)

$$\left(\frac{\delta f\,(\boldsymbol{x}, \boldsymbol{v})}{\delta t}\right)_C = \mathop{}\limits^{\cdots}_{\cdots} \left[f(\boldsymbol{x}, \boldsymbol{v}', t)\, f(\boldsymbol{x}, \boldsymbol{v}_1', t) - f(\boldsymbol{x}, \boldsymbol{v}, t)\, f(\boldsymbol{x}, \boldsymbol{v}_1, t) \right] \sigma_{\boldsymbol{v}, \boldsymbol{v}_1 \rightarrow \boldsymbol{v}', \boldsymbol{v}_1'}(\Omega)\, |\, \boldsymbol{v} - \boldsymbol{v}_1 \,|\, d\Omega\, d^3 v_1$$